MATHEMATICS

PAPER—I

Time Allowed : Three Hours

Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions

There are EIGHT questions in all, out of which FIVE are to be attempted.

Question Nos. **1** and **5** are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B.

The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

Answers must be written in ENGLISH only.

SECTION—A

1. *(a)* Let $T: R^3 \to R^3$ be a linear operator on R^3 defined by $T(x, y, z) = (2y+z, x-4y, 3x)$. Find the matrix of T in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 8

(b) The eigenvalues of a real symmetric matrix A are -1, 1 and -2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1\ 1\ 0)^T$, $(0\ 0\ 1)^T$ and $\frac{1}{\sqrt{2}}(-1\ -1\ 0)^T$ respectively. Find the matrix A^4. 8

(c) Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by xy-plane. 8

(d) Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3-3x+k=0$ has two distinct solutions in the interval $[-1, 1]$. 8

(e) If the coordinates of the points A and B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the line joining A and B is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then show that $x\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=0$. 8

2. *(a)* Determine the extreme values of the function $f(x, y)=3x^2-6x+2y^2-4y$ in the region $\{(x, y)\in\mathbb{R}^2 : 3x^2+2y^2\le 20\}$. 10

(b) Consider the singular matrix

$$A=\begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1\ 1\ 0\ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4+pA^3+qA^2+rA=0$. 15

(c) A line makes angles α, β, γ, δ with the four diagonals of a cube. Show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$ 15

3. *(a)* Consider the vectors $x_1 = (1, 2, 1, -1)$, $x_2 = (2, 4, 1, 1)$, $x_3 = (-1, -2, 0, -2)$ and $x_4 = (3, 6, 2, 0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1, x_2, x_3, x_4\}$ is a subspace of $\mathbb{R}^4$, defined as

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : 2\xi_1 - \xi_2 = 0, \;\; 2\xi_1 - 3\xi_3 - \xi_4 = 0\}$$

Can this subspace be written as $\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathbb{R}\}$? What is the dimension of this subspace? 15

(b) The dimensions of a rectangular box are linear functions of time—$l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and the height is decreasing at the rate 3 cm/sec, find the rates at which the volume V and the surface area S are changing with respect to time. If $l(0) = 10$, $w(0) = 8$ and $h(0) = 20$, is V increasing or decreasing, when $t = 5$ sec ? What about S, when $t = 5$ sec ? 10

(c) Show that the shortest distance between the straight lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{and} \quad \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

is $3\sqrt{30}$. Find also the equation of the line of shortest distance. 15

4. *(a)* Using elementary row operations, reduce the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX = b$, where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$. 15

(b) Find the centroid of the solid generated by revolving the upper half of the cardioid $r = a(1 + \cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform. 10

(c) A variable plane is parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes at the points A, B and C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$ 15

SECTION—B

5. *(a)* Solve the differential equation $(D^2+1)y = x^2 \sin 2x$; $D \equiv \frac{d}{dx}$. 8

(b) Solve the differential equation $(px - y)(py + x) = h^2 p$, where $p = y'$. 8

(c) A 2 metres rod has a weight of 2 N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3 N, 7 N and 10 N respectively. Find the point at which the rod must be supported if it is to remain horizontal. 8

(d) Let $\bar{r} = \bar{r}(s)$ represent a space curve. Find $\frac{d^3\bar{r}}{ds^3}$ in terms of $\overline{T}$, $\overline{N}$ and $\overline{B}$, where $\overline{T}$, $\overline{N}$ and $\overline{B}$ represent tangent, principal normal and binormal respectively. Compute $\frac{d\bar{r}}{ds} \cdot \left(\frac{d^2\bar{r}}{ds^2} \times \frac{d^3\bar{r}}{ds^3} \right)$ in terms of radius of curvature and the torsion. 8

(e) Evaluate $\int_{(0,\ 0)}^{(2,\ 1)} (10x^4 - 2xy^3)dx - 3x^2y^2 dy$ along the path $x^4 - 6xy^3 = 4y^2$. 8

6. *(a)* Solve by the method of variation of parameters the differential equation

$$x''(t) - \frac{2x(t)}{t^2} = t, \quad \text{where } 0 < t < \infty$$

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(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force). 15

(c) Verify Stokes' theorem for $\overline{V} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 10

7. *(a)* Find the general solution of the differential equation

$$\ddot{x} + 4x = \sin^2 2t$$

Hence find the particular solution satisfying the conditions

$$x\left(\frac{\pi}{8}\right) = 0 \quad \text{and} \quad \dot{x}\left(\frac{\pi}{8}\right) = 0$$

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(b) A vessel is in the shape of a hollow hemisphere surmounted by a cone held with the axis vertical and vertex uppermost. If it is filled with a liquid so as to submerge half the axis of the cone in the liquid and height of the cone be double the radius (r) of its base, find the resultant downward thrust of the liquid on the vessel in terms of the radius of the hemisphere and density (ρ) of the liquid. 15

(c) Derive the Frenet-Serret formulae. Verify the same for the space curve $x = 3\cos t,\ y = 3\sin t,\ z = 4t$. 10

8. (a) Find the general solution of the differential equation

$$(x-2)y'' - (4x-7)y' + (4x-6)y = 0$$ 10

(b) A shot projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h metres above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection. 15

(c) Derive $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ in spherical coordinates. 15

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