

MATHEMATICS

Paper – I

Time Allowed : Three Hours

Maximum Marks : 200

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are EIGHT questions in all, out of which FIVE are to be attempted.

Questions no. 1 and 5 are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in ENGLISH only.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be given by

$$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z).$$

Find the matrix of T with respect to standard basis of \mathbb{R}^3 and \mathbb{R}^4 (i.e., $(1, 0, 0), (0, 1, 0), \text{etc.}$). Examine if T is a linear map. 8

(b) Show that $\frac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$. 8

(c) Examine if the function $f(x, y) = \frac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at $(0, 0)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at points other than origin. 8

- (d) If the point (2, 3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord. 8

- (e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$. 8

- Q2.** (a) After changing the order of integration of $\int_0^{\infty} \int_0^{\infty} e^{-xy} \sin nx \, dx \, dy$,

show that $\int_0^{\infty} \frac{\sin nx}{x} \, dx = \frac{\pi}{2}$. 10

- (b) A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. 10

- (c) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve. 10

- (d) Let T be a linear map such that $T : V_3 \rightarrow V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2 . Find the matrix of T relative to these basis.

Further take two other basis $B_1[(1, 1, 0) (1, 0, 1) (0, 1, 1)]$ and $B_2[(1, 1) (1, -1)]$. Obtain the matrix T_1 relative to B_1 and B_2 . 10

- Q3.** (a) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$. Hence find A^{-1} . 10
- (b) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point $(1, 0, 0)$. 10
- (c) Obtain the area between the curve $r = 3(\sec \theta + \cos \theta)$ and its asymptote $x = 3$. 10
- (d) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle. 10
- Q4.** (a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. 10
- (b) Show that the integral $\int_0^{\infty} e^{-x} x^{\alpha-1} dx$, $\alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$. 10
- (c) Prove that $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2})$. 10
- (d) A plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. 10

SECTION B

- Q5.** (a) Obtain the curve which passes through (1, 2) and has a slope = $\frac{-2xy}{x^2 + 1}$.
Obtain one asymptote to the curve. 8
- (b) Solve the dE to get the particular integral of $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$. 8
- (c) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings. 8
- (d) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot \beta = 2 \tan (\alpha - \beta)$. 8
- (e) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iiint_S \bar{F} \cdot d\bar{S}$ where S is the surface bounding the volume E and $\bar{F} = (zx \sin yz + x^3) \hat{i} + \cos yz \hat{j} + (3zy^2 - e^{\lambda^2 + y^2}) \hat{k}$. 8
- Q6.** (a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet? 10
- (b) Using the method of variation of parameters, solve $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x$. 10
- (c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line. 10
- (d) Evaluate $\iint_S (\nabla \times \bar{f}) \cdot \hat{n} dS$ for $\bar{f} = (2x - y) \hat{i} - yz^2 \hat{j} - y^2z \hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane. 10

- Q7.** (a) State Stokes' theorem. Verify the Stokes' theorem for the function $\vec{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one. 15
- (b) A uniform rod of weight W is resting against an equally rough horizon and a wall, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W \tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α ? 15
- (c) Obtain the singular solution of the differential equation 10
- $$y^2 - 2pxy + p^2(x^2 - 1) = m^2, \quad p = \frac{dy}{dx}.$$
- Q8.** (a) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight $2P$. Prove that the density of the body and the liquid are in the ratio $3 : 2$. 10
- (b) Solve the differential equation 10
- $$\frac{dy}{dx} - y = y^2(\sin x + \cos x).$$
- (c) Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, if and only if either $\vec{b} = \vec{0}$ or \vec{c} is collinear with \vec{a} or \vec{b} is perpendicular to both \vec{a} and \vec{c} . 10
- (d) A particle is acted on a force parallel to the axis of y whose acceleration is λy , initially projected with a velocity $a\sqrt{\lambda}$ parallel to x -axis at the point where $y = a$. Prove that it will describe a catenary. 10

