HS/XII/A. Sc. Com/M/23

2023

MATHEMATICS

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

General Instructions :

(i) All questions are compulsory.

(ii) This question paper contains 36 questions divided into four Sections—A, B, C and D. Section—A comprises of 20 questions of *1* mark each, Section—B comprises of 6 questions of *2* marks each, Section—C comprises of 6 questions of *4* marks each and Section—D comprises of 4 questions of *6* marks each.

(iii) There is no overall choice. However, internal choice has been provided in 9 questions of Section—A, 5 questions of Section—B, 5 questions of Section—C and 3 questions of Section—D. You have to attempt only one of the alternatives in all such questions.

(iv) Use of calculator is not permitted.

SECTION—A

1. Define an equivalence relation. 1

Or

A relation R in the set N of natural numbers is defined as $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$. Find the range of R. 1

2. Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$. 1

Or

Evaluate : 1

$$\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$$

3. Find the value of x, if

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$$ 1

Or

If $A=\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$, find AB and BA. 1

4. What is an objective function of a linear programming problem? 1

5. Evaluate : 1

$$\int_0^{\pi/2} \cos 2x\, dx$$

Or

Evaluate : 1

$$\int_0^1 \frac{dx}{1+x^2}$$

6. Differentiate : 1

$$2\sqrt{\cot x}$$

Or

Find $\frac{dy}{dx}$, if $y = \sec(\tan x)$. 1

7. Write the order and degree of the differential equation

$$\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$$ 1

8. Prove that $y = Ax$ is a solution of the differential equation

$$xy' = y, \quad (x \neq 0)$$

and A is a constant. 1

9. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective, where R is the set of real numbers. 1

Or

Show that the modulus function $f : R \to R$ given by $f(x) = |x|$ is not one-one, where R is the set of real numbers. 1

10. Find $\frac{dy}{dx}$, if $y = \log(\cos e^x)$. 1

Or

If $2x + 3y = \sin x$, find $\frac{dy}{dx}$. 1

11. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find $P(A \cap B)$. 1

12. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent? 1

13. Evaluate : 1

$$\int x^2 \left(1 - \frac{1}{x^2}\right) dx$$

Or

Evaluate : 1

$$\int \tan^2 x \, dx$$

14. Compute the magnitude of the following vector : 1

$$\vec{a} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

15. Evaluate : 1

$$\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$$

16. Prove that the function

$$f(x) = x^3 - 3x^2 + 3x - 100$$

is increasing in R, where R is the set of real numbers. 1

Choose the correct answer :

17. If $\sin^{-1} x = y,$ then

(a) $0 \leq y \leq \pi$

(b) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(c) $0 < y < \pi$

(d) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ 1

18. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then $A + A' = I$, if the value of α is

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{3}$

(c) π

(d) $\frac{3\pi}{2}$ 1

19. The anti-derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ is equal to

(a) $\frac{1}{3}x^{\frac{1}{3}}+2x^{\frac{1}{2}}+c$

(b) $\frac{2}{3}x^{\frac{2}{3}}+\frac{1}{2}x^{2}+c$

(c) $\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+c$

(d) $\frac{3}{2}x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}+c$ 1

Or

The rate of change of the area of a circle with respect to its radius r at $r=6$ cm is

(a) 10π

(b) 12π

(c) 8π

(d) 11π 1

20. The value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is

(a) 0

(b) -1

(c) 1

(d) 3 1

SECTION—B

21. Find x and y, if

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$ 2

Or

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, show that $|2A| = 4|A|$. 2

22. Find the value of k, if the function defined by

$$f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$. 2

Or

Show that $f(x) = 5x - 3$ is continuous at $x = 5$. 2

23. Evaluate : 2

$$\int x e^x dx$$

Or

Evaluate : 2

$$\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$$

24. If $y = 500e^{7x} + 600e^{-7x}$, show that

$$\frac{d^2y}{dx^2} = 49y$$ 2

25. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$. 2

Or

Show that the points $A(2, 3, 4)$, $B(-1, -2, 1)$ and $C(5, 8, 7)$ are collinear. 2

26. Find $\frac{dy}{dx}$, if

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \quad 0 < x < 1$$ 2

Or

Evaluate : 2

$$\int \frac{x^2}{1-x^6}\, dx$$

SECTION—C

27. Show that the function $f : R \to R$ defined by $f(x) = 3 - 4x$ is one-one and onto, where R is the set of real numbers. 4

Or

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. 4

28. Find the intervals in which the function f given by

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

is *(a)* strictly increasing and *(b)* strictly decreasing. 4

Or

Sand is pouring from a pipe at the rate of $12 \text{ cm}^3/\text{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? 4

29. Using the properties of definite integrals, evaluate

$$\int_0^{\pi/4} \log(1 + \tan x)\, dx$$ 4

Or

Using partial fraction, evaluate

$$\int \frac{1}{x^4 - 1}\, dx$$ 4

30. Solve the following homogeneous differential equation : 4

$$xdy - ydx = \sqrt{x^2 + y^2}\, dx$$

Or

Solve the following linear differential equation :

$$(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}$$

given $y = 0$, when $x = 1$. 4

31. If a unit vector $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence the components of $\vec{a}$. 4

Or

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. 4

32. Solve the following linear programming problem graphically : 4

Maximize $Z = 4x + y$

subject to the constraints

$$x + y \le 50$$

$$3x + y \le 90$$

$$x \ge 0,\ y \ge 0$$

SECTION—D

33. Verify $A(\text{adj } A) = (\text{adj } A)A = |A|\, I$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$ 6

Or

Solve the following system of linear equations using matrix method : 6

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

34. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere. 6

Or

Using definite integral, find the area of the region bounded by the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$ 6

35. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

and

$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$ 6

36. A manufacturer has three machine operators *A*, *B* and *C*. The first operator *A* produces 1% defective items whereas the other two operators *B* and *C* produce 5% and 7% defective items respectively. *A* is on the job for 50% of the time, *B* is on the job for 30% of the time and *C* is on the job for 20% of the time. A defective item is produced. What is the probability that it was produced by *A*? 6

Or

Let *X* denote the number of hours you study during a randomly selected school day. The probability that *X* can take the values *x* has the following form, where *k* is a constant :

$$P(X = x) = \begin{cases} 0\cdot 1 & , \text{ if } x = 0 \\ kx & , \text{ if } x = 1 \text{ or } 2 \\ k(5 - x) & , \text{ if } x = 3 \text{ or } 4 \\ 0 & , \text{ otherwise} \end{cases}$$

Find the value of *k*. What is the probability that you study at least 2 hours, exactly 2 hours and at most 2 hours? 6

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