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MATHEMATICS

Full Marks : 100

Time : 3 hours

General Instructions :

 (i) Write all the answers in the Answer Script.

 (ii) The question paper consists of three Sections—A, B and C.

(iii) Section—A consists of 15 questions, carrying *2* marks each.

(iv) Section—B consists of 10 questions, carrying *4* marks each, out of which 2 questions have internal choices.

 (v) Section—C has 5 questions, carrying *6* marks each, out of which 2 questions have internal choices.

SECTION—A

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is one-one and onto.

2. Evaluate :

$$\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$$

(2)

3. If $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, then find $A - 4B$.

4. Express the matrix $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

5. For what value of k the function

$$f(x) = \frac{k \cos x}{\pi - 2x}, \quad \text{when } x \neq \frac{\pi}{2}$$
$$= 3 \qquad , \quad \text{when } x = \frac{\pi}{2}$$

is continuous at $x = \frac{\pi}{2}$?

6. Show that

$$\frac{d}{dx}(\sin 2x \sin 4x) = 3 \sin 6x - \sin 2x$$

7. The side of a square sheet of metal is increasing at 3 cm per minute. At what rate the area is increasing when the side is 10 cm long?

8. Evaluate :

$$\int_{-\pi/2}^{\pi/2} |\sin x| \, dx$$

9. Solve the equation

$$\log\left(\frac{dy}{dx}\right) = ax + by$$

10. Evaluate :

$$\int e^x \left(\frac{1 + x\log x}{x} \right) dx$$

11. Find the value of the integral

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

12. Find the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ where

$$\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k} \text{ and } \vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$$

13. Find the value of k so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2k} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3k} = \frac{5-y}{1} = \frac{6-z}{5}$$

are at right angles.

14. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, find the value of

$$\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$$

15. The probability that a student selected at random from a class will pass in Hindi is $\frac{4}{5}$ and the probability that he passes in Hindi and English is $\frac{1}{2}$. What is the probability that he will pass in English if it is known that the student has passed in Hindi?

SECTION—B

16. Let $\mathbb{N}$ be the set of natural numbers and let R be a relation on $\mathbb{N} \times \mathbb{N}$ defined by

$$(a, b)R(c, d) \Leftrightarrow ad = bc$$

Prove that R is an equivalence relation.

17. Using the properties of determinants, show that

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

18. If $x = \dfrac{1 + \log t}{t^2}$ and $y = \dfrac{3 + 2\log t}{t}$, then show that $\dfrac{dy}{dx} = t.$

19. Evaluate :

$$\int (\sin^{-1} x)^2 \, dx$$

20. Using differential, find the approximate value of $\sqrt[3]{127}$.

21. Find the interval in which the function $f(x) = 5 + 36x + 3x^2 - 2x^3$ is *(a)* strictly increasing and *(b)* strictly decreasing.

Or

Find the equation of the tangent to the curve $x^2 + 3y = 3$ which is parallel to the line $y - 4x + 5 = 0$.

22. Using the properties of definite integral, show that

$$\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}} \, dx = \frac{\pi}{12}$$

23. Evaluate the following integral as the limit of a sum :

$$\int_{1}^{4} (3x^2 + 2x) \, dx$$

24. Find the equation of the plane passing through the point $(1, 0, -2)$ and perpendicular to each of the planes $2x + y - z - 2 = 0$ and $x - y - z - 3 = 0$.

Or

Find the length and foot of the perpendicular from the point $(7, 14, 5)$ to the plane $2x + 4y - z = 2$.

25. Find the shortest distance between the lines

$$\vec{r} = (6\hat{i} + 3\hat{k}) + \lambda (2\hat{i} - \hat{j} + 4\hat{k}) \text{ and}$$
$$\vec{r} = (-9\hat{i} + \hat{j} - 10\hat{k}) + \mu (4\hat{i} + \hat{j} + 6\hat{k})$$

SECTION—C

26. Using integration, find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

27. Using matrices, solve the following system of equations :

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$$
$$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$$

28. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $(500\ \pi)$ cm^3.

Or

A window is in the form of a rectangle, surmounted by a semicircular opening. The total perimeter of the window is 10 metres. Find the dimensions of the window to admit maximum light through it.

29. A box contains 16 bulbs, out of which 4 bulbs are defective. 3 bulbs are drawn one by one from the box without replacement. Let X be the number of defective bulbs drawn. Find the mean and variance of X.

30. A manufacturer produces two types of steel trunks. He has two machines *A* and *B*. The first type of trunk requires 3 hours on machine *A* and 3 hours on machine *B*. The second type requires 3 hours on machine *A* and 2 hours on machine *B*. Machines *A* and *B* can work at most 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 and ₹ 25 per trunk of first and second type respectively. How many trunks of each type must he make each day to make maximum profit?

Or

Two tailors *A* and *B*, earn ₹ 300 and ₹ 400 per day respectively. *A* can stitch 6 shirts and 4 pairs of trousers per day while *B* can stitch 10 shirts and 4 pairs of trousers per day. How many days should each of them work if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost?

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