West Bengal JEE 2020 Mathematics Question Paper.

1 Let $\varphi(x)=f(x)+f(1-x)$ and $f''(x)<0$ in $[0,1]$, then

A. $\varphi$ is monotonic increasing in $\left[0,\frac{1}{2}\right]$ and monotonic decreasing in $\left[\frac{1}{2},1\right]$

B. $\varphi$ is monotonic increasing in $\left[\frac{1}{2},1\right]$ and monotonic decreasing in $\left[0,\frac{1}{2}\right]$

C. $\varphi$ is neither increasing nor decreasing in any sub interval of $[0,1]$

D. $\varphi$ is increasing in $[0,1]$

$\phi(x)=f(x)+f(1-x)$

$\phi'(x)=f'(x)-f'(1-x)\ldots(1)$

$f''(x)<0\Rightarrow f'(x)$ is a decreasing function

case 1: $0\leq x\leq\frac{1}{2}\Rightarrow x\leq 1-x$

$\Rightarrow f'(x)\geq f'(1-x)$

$\Rightarrow f'(x)-f'(1-x)\geq 0$

From eqn (1) $\Rightarrow\phi'(x)>0\Rightarrow\phi$ is increasing in $\left[0,\frac{1}{2}\right]$

case 2: $\frac{1}{2}\leq x\leq 1\Rightarrow x\geq 1-x$

$\Rightarrow f'(x)\leq f'(1-x)$

From eqn (1) $\Rightarrow f'(x)-f'(1-x)\leq 0\Rightarrow\phi$ is decreasing in $\left[\frac{1}{2},1\right]$

2 Let $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$. Then

$$\left(\text{Here } y_2 \equiv \frac{d^2y}{dx^2}, y_1 \equiv \frac{dy}{dx}\right)$$

- A. $x^2y_2 + xy_1 + n^2y = 0$ ✔
- B. $xy_2 - xy_1 + 2n^2y = 0$ ✘
- C. $x^2y_2 + 3xy_1 - n^2y = 0$ ✘
- D. $xy_2 + 5xy_1 - 3y = 0$ ✘

$$\frac{-1}{\sqrt{1-\frac{y^2}{b^2}}} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = n \cdot \frac{n}{x} \cdot \frac{1}{n}$$

$$\frac{dy}{dx}\left(\frac{-1}{\sqrt{b^2-y^2}}\right) = \frac{n}{x}$$

$$\Rightarrow xy_1 = -n\sqrt{b^2-y^2}$$

Squaring both sides

$$x^2y_1^2 = n^2b^2 - n^2y^2$$

Differentiating both sides

$$\Rightarrow 2xy_1^2 + x^2.2y_1y_2 = -n^2.2y\, y_1$$

$$\Rightarrow xy_1 + x^2y_2 = -n^2y$$

$$\Rightarrow x^2y_2 + xy_1 + n^2y = 0$$

3 $\int \frac{f(x)\phi'(x)+\phi(x)f'(x)}{(f(x)\phi(x)+1)\sqrt{f(x)\phi(x)-1}}dx =$

A. $\sin^{-1}\sqrt{\frac{f(x)}{\phi(x)}}+c$

B. $\cos^{-1}\sqrt{(f(x))^2-(\phi(x))^2}+c$

C. $\sqrt{2}\,\tan^{-1}\sqrt{\frac{f(x)\phi(x)-1}{2}}+c$

D. $\sqrt{2}\,\tan^{-1}\sqrt{\frac{f(x)\phi(x)+1}{2}}+c$

Let

$$I=\int \frac{f(x)\phi'(x)+\phi(x)f'(x)}{(f(x)\phi(x)+1)\sqrt{f(x)\phi(x)-1}}dx$$

Let $f(x)\phi(x)-1=t^2$

$\Rightarrow f'(x)\phi(x)+f(x)\phi'(x)=2tdt$

$$\therefore I=\int \frac{2t\,dt}{(t^2+2)(t)}$$

$$\Rightarrow I=2\int \frac{dt}{t^2+2}$$

$$\Rightarrow I=\frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c$$

$$\Rightarrow I=\sqrt{2}\tan^{-1}\left(\sqrt{\frac{f(x)\phi(x)-1}{2}}\right)+c$$

4 The value of $\sum_{n=1}^{10} \int_{-2n-1}^{-2n} \sin^{27} x \, dx + \sum_{n=1}^{10} \int_{2n}^{2n+1} \sin^{27} x \, dx$ is equal to

A. $27$

B. $54$

C. $-54$

D. $0$

Let $I_1 = \sum_{n=1}^{10} \int_{-2n-1}^{-2n} \sin^{27} x \, dx,$

$I_2 = \sum_{n=1}^{10} \int_{2n}^{2n+1} \sin^{27} x \, dx$

$I_1 = \sum_{n=1}^{10} \int_{-2n-1}^{-2n} \sin^{27} x \, dx,$

Let $x = -t \Rightarrow dx = -dt$

$\Rightarrow I_1 = \sum_{n=1}^{10} \int_{2n+1}^{2n} \sin^{27}(-t)(-dt)$

$\Rightarrow I_1 = \sum_{n=1}^{10} \int_{2n+1}^{2n} \sin^{27} x \, dx$

$\Rightarrow I_1 = -\sum_{n=1}^{10} \int_{2n}^{2n+1} \sin^{27} x \, dx$

$\Rightarrow I_1 = -I_2$

$\Rightarrow I_1 + I_2 = 0$

5 $\int_0^2 [x^2]$ is equal to

A. $1$

B. $5-\sqrt{2}-\sqrt{3}$

C. $3-\sqrt{2}$

D. $\frac{8}{3}$

$$\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{2} 3dx$$

$$= (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})$$

$$= 5-\sqrt{3}-\sqrt{2}.$$

6 If the tangent to the curve $y^2 = x^3$ at $(m^2, m^3)$ is also a normal to the curve at $(M^2, M^3)$, then the value of $mM$ is

✗ A. $-\frac{1}{9}$

✗ B. $-\frac{2}{9}$

✗ C. $-\frac{1}{3}$

✓ D. $-\frac{4}{9}$

$y^2 = x^3$

$\Rightarrow 2y\frac{dy}{dx} = 3x^2$

Slope, $m = \frac{3x^2}{2y}$

Slope of tangent at $(m^2, m^3)$

$\Rightarrow m_1 = \frac{3m^4}{2m^3}$

$3m$

7 If $x^2 + y^2 = a^2$, then $\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx =$

✗ A. $2\pi a$

✗ B. $\pi a$

✓ C. $\frac{1}{2}\pi a$

✗ D. $\frac{1}{4}\pi a$

$x^2 + y^2 = a^2$

Differentiating both sides, we get:

$\frac{dy}{dx} = -\frac{x}{y}$

$\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$= \int_0^a \sqrt{1 + \frac{x^2}{y^2}} dx$

$= a\int_0^a \frac{1}{\sqrt{a^2 - x^2}} dx$

$= a\left[\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{1}{2}\pi a$

8 Let $f$, be a continuous function in $[0, 1]$, then $\lim\limits_{n\to\infty} \sum\limits_{j=0}^{n} \frac{1}{n} f\left(\frac{j}{n}\right)$ is

✗ A. $\frac{1}{2}\int\limits_{0}^{1/2} f(x)dx$

✗ B. $\int\limits_{1/2}^{1} f(x)dx$

✓ C. $\int\limits_{0}^{1} f(x)dx$

✗ D. $\int\limits_{0}^{1/2} f(x)dx$

$$\lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{n} f\left(\frac{j}{n}\right)$$

$\frac{1}{n} \to dx$, $\frac{j}{n} \to x$, upper limit, $\lim \frac{n}{n} = 1$.

9 Let $f$ be a differentiable function with $\lim\limits_{x\to\infty} f(x) = 0$. If $y' + yf'(x) - f(x)f'(x) = 0$, $\lim\limits_{x\to\infty} y(x) = 0$, then

(where $y' \equiv \frac{dy}{dx}$)

✗ A. $y + 1 = e^{f(x)} + f(x)$

✗ B. $y - 1 = e^{f(x)} + f(x)$

✓ C. $y + 1 = e^{-f(x)} + f(x)$

✗ D. $y - 1 = e^{-f(x)} + f(x)$

$$\frac{dy}{dx} + yf'(x) = f(x)f'(x)$$

$$I.F = e^{\int f'(x)dx} = e^{f(x)}$$

$$y.e^{f(x)} = \int e^{f(x)} f(x) f'(x) dx$$

Let $f(x) = t \Rightarrow f'(x)dx = dt$

$$\Rightarrow ye^{f(x)} = \int e^t t dt$$

$$\Rightarrow ye^{f(x)} = e^t(t-1) + c$$

$$\Rightarrow ye^{f(x)} = e^{f(x)}(f(x) - 1) + c$$

Given, $\lim\limits_{x\to\infty} f(x) = 0$, $\lim\limits_{x\to\infty} y(x) = 0$

$$\Rightarrow 0.e^0 = e^0(0-1) + c \Rightarrow c = 1$$

$$\therefore ye^{f(x)} = e^{f(x)}(f(x) - 1) + 1$$

$$\Rightarrow y = f(x) - 1 + e^{-f(x)}$$

$$\Rightarrow y + 1 = f(x) + e^{-f(x)}$$

10 Let $f(x) = 1 - \sqrt{(x^2)}$ where the square root is to be taken positive, then

A. $f$ has no extrema at $x = 0$

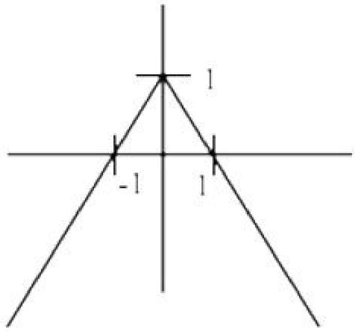
B. $f$ has minima at $x = 0$

C. $f$ has maxima at $x = 0$

D. $f'$ exists at $0$

$f(x) = 1 - \sqrt{x^2}$

$\Rightarrow f(x) = 1 - |x|$




$\therefore$ Max at $x = 0$ and is $1$

11 If $x \sin\left(\frac{y}{x}\right) dy = \left[y \sin\left(\frac{y}{x}\right) - x\right] dx, x > 0$ and $y(1) = \frac{\pi}{2}$ then the value of $\cos\left(\frac{y}{x}\right)$ is

A. $1$

B. $\log x$

C. $e$

D. $0$

$$\sin\left(\frac{y}{x}\right) \frac{dy}{dx} = \frac{y}{x} \sin \frac{y}{x} - 1 \Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x} \sin \frac{y}{x} - 1}{\sin \frac{y}{x}}$$

$$\text{Let } \frac{y}{x} = t \Rightarrow \frac{dy}{dx} = t + x \frac{dt}{dx}$$

$$\Rightarrow t + x \frac{dt}{dx} = t - \frac{1}{\sin t} \Rightarrow -\int \sin t \, dt = \int \frac{dx}{x}$$

$$\Rightarrow \cos t = \log x + c$$

$$\Rightarrow \cos \frac{y}{x} = \log x + c$$

$$y(1) = \frac{\pi}{2} \Rightarrow \cos \frac{\pi}{2} = \log 1 + c \Rightarrow c = 0$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log x$$

12 If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1[a > 0]$ attains its maximum and minimum at $p$ and $q$ respectively such that $p^2 = q$, then $a$ is equal to

A. $2$

B. $\frac{1}{2}$

C. $\frac{1}{4}$

D. $3$

$$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1, a > 0$$

$$f'(x) = 6x^2 - 18ax + 12a^2$$

$$= 6(x^2 - 3ax + 2a^2) = 0 \text{ for extreme values}$$

$$= 6(x - a)(x - 2a) = 0 \Rightarrow x = a, 2a$$

$$f''(x) = 12x - 18a, f''(a) = -6a < 0 \text{ Max at x = a = p}$$

$$f''(2a) = 6a > 0 \text{ Min at x = 2a = q}$$

$$\text{Given } p^2 = q \Rightarrow a^2 = 2a \Rightarrow a = 2$$

13 If $a$ and $b$ are arbitary positive real numbers, then the least possible value of $\frac{6a}{5b}+\frac{10b}{3a}$ is

A. $4$

B. $\frac{6}{5}$

C. $\frac{10}{3}$

D. $\frac{68}{15}$

$AM \geq GM$

$$\frac{\frac{6a}{5b}+\frac{10b}{3a}}{2} \geq \left(\frac{6a}{5b}\cdot\frac{10b}{3a}\right)^{\frac{1}{2}}$$

$$\Rightarrow \frac{6a}{5b}+\frac{10b}{3a} \geq 2\cdot 4^{\frac{1}{2}}$$

$$\Rightarrow \frac{6a}{5b}+\frac{10b}{3a} \geq 4$$

14 If $2\log(x+1)-\log(x^2-1)=\log 2$, then $x=$

A. only $3$

B. $-1$ and $3$

C. only $-1$

D. $1$ and $3$

$$\log(x+1)^2-\log(x^2-1)=\log 2$$

$$\log\left|\frac{(x+1)^2}{(x^2-1)}\right| = \log 2 \Rightarrow \frac{x+1}{x-1}=2$$

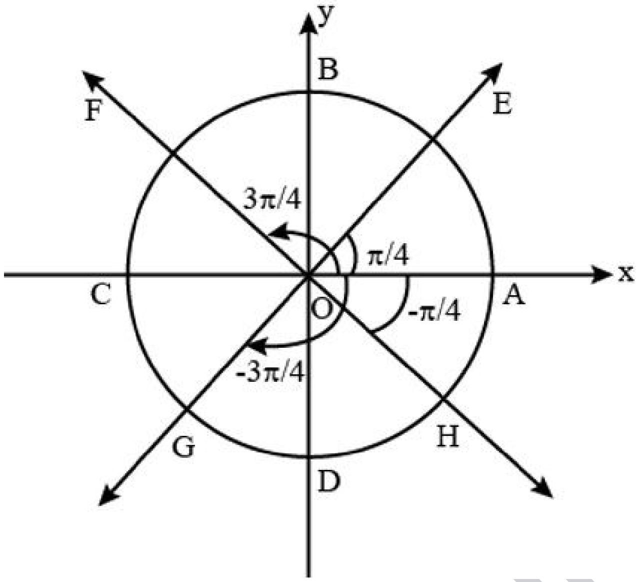
$$\Rightarrow x+1=2x-2$$

$$\Rightarrow x=3$$

15 The number of complex numbers $p$ such that $|p| = 1$ and imaginary part of $p^4$ is $0$ , is

- A. $4$
- B. $2$
- C. $8$
- D. infinitely many

$|p| = 1$, imaginary part of $p^4 = 0$
i.e circle with centre $(0, 0)$ & rad = 1




$A = (1, 0)$
$B = (0, 1)$
$C = (-1, 0)$
$D = (0 - 1)$
$E = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$
$E^4 = \cos\pi + i\sin\pi$
$F = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
$G = \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)$
$H = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$
$i.e\ 8$

16 The equation $z\bar{z}+(2-3i)z+(2+3i)\bar{z}+4=0$ represents a circle of radius

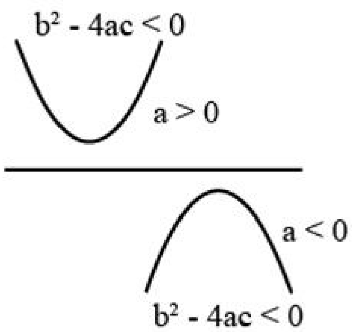
- A. 2 unit
- B. 3 unit ✓
- C. 4 unit
- D. 6 unit

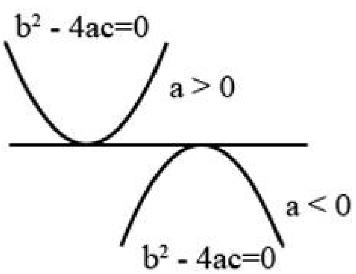
$$z\bar{z}+(2-3i)z+(2+3i)\bar{z}+4=0$$

$$\begin{aligned}\text{radius } &= \sqrt{(2-3i)(2+3i)-4}\\ &= \sqrt{9}\\ &= 3\end{aligned}$$

17 The expression $ax^2+bx+c$ ($a, b$ and $c$ are real) has the same sign as that of a for all $x$ is

- A. $b^2-4ac>0$
- B. $b^2-4ac\neq 0$
- C. $b^2-4ac\leq 0$ ✓
- D. $b$ and $c$ have the same sign as that of $a$



18 In a $12$ storied building , $3$ person enter a lift cabin. It is known that they will leave the lift at different floors. In how many ways can they do so if the lift does not stop at the second floor?

A. $36$

B. $120$

C. $240$

D. $720$ ✓

The lift can stop at $12-1-1=10$ floors
So total required ways is $=10\times9\times8=720$

19 If the total number of $m$-element subsets of the set $A=\{a_1,a_2,\ldots,a_n\}$ is $k$ times the number of $m$ element subsets containing $a_4$ then $n$ is

A. $(m-1)k$

B. $mk$ ✓

C. $(m+1)k$

D. $(m+2)k$

From set of $n$ element selecting a subset of $m$ element
$={}^nC_m$

Now, $a_4$ is already selected.
$\therefore$ Total number of sets which contains $a_4$ is ${}^{n-1}C_{m-1}$.

Now, it is given that
${}^nC_m=k\cdot{}^{n-1}C_{m-1}$

$$\Rightarrow\frac{n!}{m!(n-m)!}=k\cdot\frac{(n-1)!}{(m-1)!(n-m)!}$$
$\Rightarrow n=mk$

20 Let $I(n) = n^n, J(n) = 1.3.5....(2n-1)$ for all $(n > 1), n \in N$, then

A. $I(n) > J(n)$

B. $I(n) < J(n)$

C. $I(n) \neq J(n)$

D. $I(n) = \frac{1}{2} J(n)$

$$\frac{I(n)}{J(n)} = \frac{n \cdot n \cdot n \cdot n \cdots n}{1 \cdot 3 \cdot 5 \cdot n \cdots 2n-1}$$

or

$$= \frac{n \cdot 2 \cdot n \cdot 4 \cdot n \cdot 6 \cdot n \cdots 2n \cdot n}{1 \cdot 2 \cdot 3 \cdot n \cdots 2n}$$

$$= \frac{2^n n! \cdot n^n}{(2n)!}$$

$$= \frac{2^n \cdot n^n}{{}^{2n}P_n} > 1$$

21 If $c_0, c_1, c_2, \cdots c_{15}$ are the Binomial co-efficients in the expansion of $(1+x)^{15}$, then the value of $\frac{c_1}{c_0} + 2\frac{c_2}{c_1} + 3\frac{c_3}{c_2} + \cdots + 15\frac{c_{15}}{c_{14}}$ is

A. 1240

B. 120

C. 124

D. 140

$$\frac{c_1}{c_0} + 2\frac{c_2}{c_1} + 3\frac{c_3}{c_2} + \cdots + 15\frac{c_{15}}{c_{14}}$$

$$= 15 + 2\left(\frac{15-1}{2}\right) + 3\left(\frac{15-2}{3}\right) + \ldots + 15\left(\frac{15-14}{15}\right)$$

$$\left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$= 15 + 14 + 13 + \cdots + 1 = \frac{15 \times 16}{2} = 120$$

23 Let $A=\begin{pmatrix}12 & 24 & 5\\ x & 6 & 2\\ -1 & -2 & 3\end{pmatrix}$. The value of $x$ for which the matrix $A$ is not invertible is

- A. $6$
- B. $12$
- C. $3$ ✔
- D. $2$

For matrix to be non invertible $\det A=0$

$$\begin{pmatrix}12 & 24 & 5\\ x & 6 & 2\\ -1 & -2 & 3\end{pmatrix}$$

$$|A|=2\begin{vmatrix}12 & 12 & 5\\ x & 3 & 2\\ -1 & -1 & 3\end{vmatrix}=0$$

$\Rightarrow x=3\cdots(C_1\ \&\ C_2$ are identical$)$

24 Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ be a $2\times 2$ real matrix with $\det A=1$. If the equation det $(A-\lambda I_2)=0$ has imaginary roots ($I_2$ be the Identify matrix of order $2$), then

- A. $(a+d)^2<4$ ✔
- B. $(a+d)^2=4$
- C. $(a+d)^2>4$
- D. $(a+d)^2=16$

$\det A=1$
$ad-cb=1$
Now

$$A-\lambda I_2=\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix}$$

$\Rightarrow \det(A-\lambda I_2)=0$

$\lambda^2-\lambda(a+d)+ad-cb=0$

$\because$ roots are imaginary
$\therefore D<0$

$(a+d)^2-4(ad-cb)<0$

$(a+d)^2<4$

25 If $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix} = ka^2b^2c^2$, then $k =$

A. $2$

B. $-2$

C. $-4$

D. $4$ ✓

Let $a=1, b=1, c=1$
(without losss of generelity)
$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = k$
$1(1-2)-(2-1)+2(4-1)=k$
$-1-1+6=k$
$k=4$

26 If $f: S \to R$ where $S$ is the set of all non-singular matrices of order $2$ over $R$ and $f\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = ad-bc$, then

A. $f$ is bijective mapping

B. $f$ is one-one but not onto

C. $f$ is onto but not one-one

D. $f$ is neither one-one nor onto ✓

$f\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = ad-bc$

$f\left[\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\right] = 10-12=-2$

$f\left[\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right] = 0-2=-2$

$\therefore$ Not one-one funtion
As the matrix is non singular matrix
So pre-image of zero doesn't exist
$\therefore$ Not onto

27 Let the relation $\rho$ be defined on $R$ by $a \rho b$ holds if and only if $a-b$ is zero or irrational then

A. $\rho$ is equivalence relation

B. $\rho$ is reflextive & symmetric but is not transitive

C. $\rho$ is reflexive and transitive but is not symmetric

D. $\rho$ is reflexive only

For $\rho$ to be reflexive
Taking element $(a, a)$
$a \rho a \Rightarrow a-a=0$
So $\rho$ is reflexive
For $\rho$ to be symmetric
Taking element $(a, b)$ such that $a \neq b$
If $a-b$ irrational
$\Rightarrow b-a$ irrational
So $\rho$ is symmetric
For $\rho$ to be transitive
Let $1, \sqrt{2}, 2 \in R$ such that
$1-\sqrt{2}=\text{irrational}; \sqrt{2}-2=\text{irrational}$
$\Rightarrow a-c=\text{rational}$
So $\rho$ is not transitive

28 The unit vector in $ZOX$ plane, making angles $45^\circ$ and $60^\circ$ respectively with $\vec{\alpha} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{\beta} = \hat{j} - \hat{k}$ is

A. $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

B. $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}$

C. $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

D. $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$

Vector in $ZOX$ plane is $a\hat{i} + c\hat{k}$

$\therefore \frac{(2\hat{i} + 2\hat{j} - \hat{k}).(a\hat{i} + c\hat{k})}{3\sqrt{a^2 + c^2}} = \frac{1}{\sqrt{2}}$

and $\frac{(\hat{j} - \hat{k}).(a\hat{i} + c\hat{k})}{\sqrt{2}\sqrt{a^2 + c^2}} = \frac{1}{2}$

Solve to get $a = \frac{1}{\sqrt{2}}, c = \frac{-1}{\sqrt{2}}$

(or)

checking the options

$\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{k}}{\sqrt{2}}.$

29 Four persons $A, B, C$ and $D$ throw an unbiased die, turn by turn, in succession till one gets an even numbers and win the game. What is the probability that $A$ wins if $A$ begins?

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $\frac{7}{12}$

D. $\frac{8}{15}$ ✔

$A$ wins in $1^{st}$ attempt $P(\text{even number}) = \frac{1}{2}$, $P(\text{odd number}) = \frac{1}{2}$

$$\Rightarrow P(A) + P(\bar{A})P(B)P(\bar{C})P(D)P(A) + \ldots$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots$$

$$\Rightarrow \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^9 + \cdots,$$

$$S_\infty = \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)^4}$$

$$= \frac{\frac{16}{2}}{16 - 1} = \frac{8}{15}$$

30 The rifleman is firing at a distant target and has only $10\%$ chance of hitting it. The least number of rounds he must fire to have more than $50\%$ chance of hitting it at lease once is

A. $5$

B. $7$

C. $9$

D. $11$

$p=\frac{1}{10}, q=\frac{9}{10}$

$A \rightarrow$ he hits the target

$A' \rightarrow$ Not hitting the target

$P(A')=q \cdot q \cdot q \cdots q(\text{say } n \text{ times})$

now $P(A)+P(A')=1$

$P(A)=1-P(A')$

$=1-q^n$

But $P(A)>\frac{1}{2} \Rightarrow 1-q^n>\frac{1}{2}$

$\Rightarrow \frac{1}{2}>\left(\frac{9}{10}\right)^n$

Now $n=6 \Rightarrow .5314$

$n=7 \Rightarrow .47$

$\therefore$ Requires atleast 7 shots.

31 $\cos(2x+7) = a(2-\sin x)$ can have a real solution for

- A. all real values of $a$
- B. $a \in [2, 6]$
- C. $a \in (-\infty, 2) \setminus \{0\}$
- D. $a \in (0, \infty)$

Note: There is an error in this question.
The correct equation is $\cos 2x + 7 = a(2-\sin x)$

$\cos 2x + 7 = a(2-\sin x)$
$\Rightarrow 1 - 2\sin^2 x + 7 = 2a - a\sin x$
$\Rightarrow 2\sin^2 x - a\sin x + 2a - 8$
$\Rightarrow \sin x = \dfrac{a \pm \sqrt{a^2 - 8(2a-8)}}{4}$
$\Rightarrow \sin x = \dfrac{a \pm (a-8)}{4}$
$\Rightarrow \sin x = \dfrac{a-4}{2}$

We know that,
$-1 \le \sin x \le 1$
$\Rightarrow -1 \le \dfrac{a-4}{2} \le 1$
$\Rightarrow 2 \le a \le 6$

32 The differential equation of the family of curves $y = e^x(A \cos x + B \sin x)$ where $A, B$ are arbitary constants is

A. $\frac{d^2y}{dx^2} - 9x = 13$

B. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

C. $\frac{d^2y}{dx^2} + 3y = 4$

D. $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - xy = 0$

$y = e^x(A\cos x + B\sin x)$
$ye^{-x} = A\cos x + B\sin x$
$-ye^{-x} + y_1e^{-x} = -A\sin x + B\cos x$
$-y_1e^{-x} + ye^{-x} + y_2e^{-x} - y_1e^{-x} = -A\cos x - B\sin x$
$y_2e^{-x} - 2y_1e^{-x} + ye^{-x} = -ye^{-x}$
$\Rightarrow y_2 - 2y_1 + 2y = 0$

33 The equation $r\cos\left(\theta - \frac{\pi}{3}\right) = 2$ represents

A. a circle

B. a parabola

C. an ellipse

D. a straight line

$r\cos\theta \cdot \frac{1}{2} + r\sin\theta \cdot \frac{\sqrt{3}}{2} = 2$
$r\cos\theta + \sqrt{3}r\sin\theta = 4$
$\frac{r\cos\theta}{4} + \frac{r\sin\theta}{\frac{4}{\sqrt{3}}} = 1$

34 The locus of the centre of the circles which touch both the circles $x^2+y^2=a^2$ and $x^2+y^2=4ax$ externally is

A. a circle

B. a parabola

C. an ellipse

D. a hyperbola

let $C_1 := x^2+y^2=a^2$
$C_2 : (x-2a)^2+y^2=4a^2$
So any point $P$ which moving such that distance from one point is constant distance for the $C_1(0,0)$ & $C_2(2a,0)$
and $\frac{PC_2}{PC_1}=\frac{r+2a}{r+a}>1$ (where r is radius of the required circle)
So path will be hyperbola

35 Let each of the equations $x^2+2xy+ay^2=0$ & $ax^2+2xy+y^2=0$ represent two straight lines passing through the origin. If they have a common line, then the other two lines are given by

A. $x-y=0, x-3y=0$

B. $x+3y=0, 3x+y=0$

C. $3x+y=0, 3x-y=0$

D. $(3x-2y)=0, x+y=0$

Let $y=mx$ be a line common to the given pairs of lines. Then
$am^2+2m+1=0\cdots(1)$
and $m^2+2m+a=0\cdots(2)$
Solving the above equations, we have
$\Rightarrow m^2=1$ and $m=-\frac{a+1}{2}$
$\Rightarrow (a+1)^2=4 \Rightarrow a=1,-3$
But for $a=1$, the two pairs have both the lines common. So $a=-3$ and the slope m of the line common to both the pairs is $1$.
Now
$x^2+2xy+ay^2=x^2+2xy-3y^2=(x-y)(x+3y)$
and
$ax^2+2xy+y^2=-3x^2+2xy+y^2=-(x-y)(3x+y)$
So the equation of the required equation of the line is
$3x+y=0; x+3y=0$

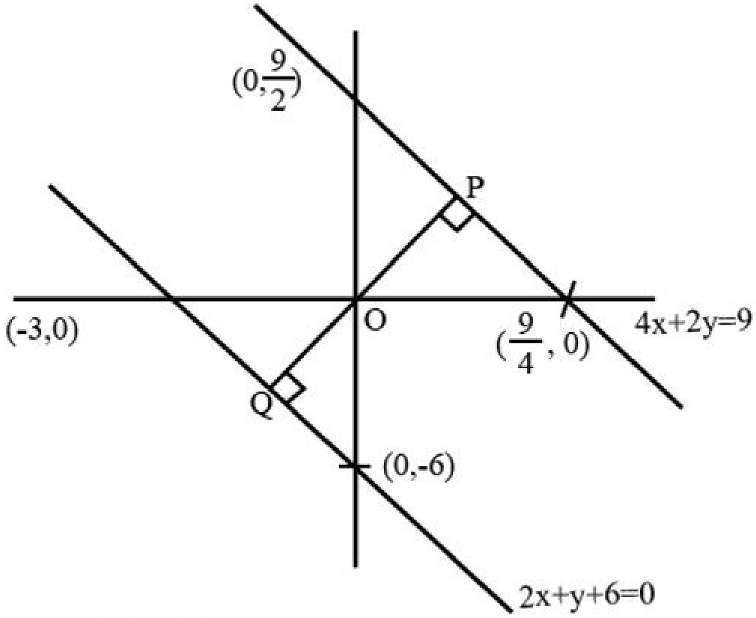
36 A straight line through the origin $O$ meets the parallel lines $4x+2y=9$ and $2x+y+6=0$ at $P$ and $Q$ respectively. the point $O$ divides the segment $PQ$ in the ratio

A. $1:2$

B. $3:4$

C. $2:1$

D. $4:3$




$$OP=\left|\frac{0-9}{\sqrt{16+4}}\right|=\frac{9}{2\sqrt{5}}$$

$$OQ=\frac{6}{\sqrt{5}}$$

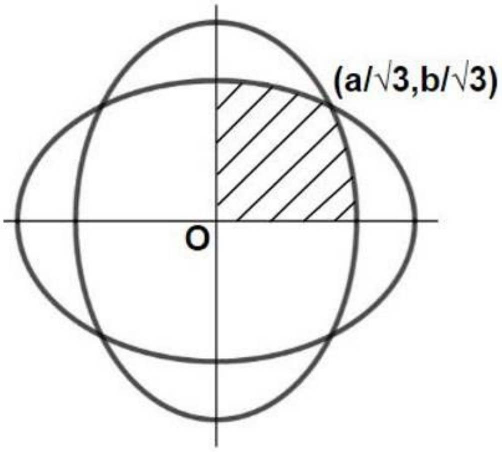
37 Area in the first quadrant between the ellipses $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$ is

A. $\frac{a^2}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$

B. $\frac{3a^2}{4}\tan^{-1}\frac{1}{2}$

C. $\frac{5a^2}{2}\sin^{-1}\frac{1}{2}$

D. $\frac{9\pi a^2}{2}$




Point of intersection of $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$ is $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

Required area is,

$$A = \frac{1}{\sqrt{2}}\int_0^{a/\sqrt{3}} \sqrt{a^2 - x^2}\,dx + \int_{a/\sqrt{3}}^{a/\sqrt{2}} \sqrt{a^2 - 2x^2}\,dx$$

$$= \frac{1}{\sqrt{2}}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^{a/\sqrt{3}}$$

$$+ \left[\frac{x}{2}\sqrt{a^2 - 2x^2} + \frac{a^2}{2\sqrt{2}}\sin^{-1}\frac{\sqrt{2}x}{a}\right]_{a/\sqrt{3}}^{a/\sqrt{2}}$$

$$= \frac{a^2}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

38 The equation of circle of radius $\sqrt{17}$ unit, with centre on the positive side of $x$-axis and through the point $(0, 1)$ is

A. $x^2 + y^2 - 8x - 1 = 0$

B. $x^2 + y^2 + 8x - 1 = 0$

C. $x^2 + y^2 - 9y + 1 = 0$

D. $2x^2 + 2y^2 - 3x + 2y = 4$

Radius = $\sqrt{17}$
Centre on positive side of x-axis i.e., $(\alpha, 0)$

$\therefore$ Equation of circle in $(x - \alpha)^2 + y^2 = (\sqrt{17})^2$

Since, it passes through $(0, 1)$
$\therefore \alpha^2 + 1^2 = 17$
$\Rightarrow \alpha^2 = 16$
$\Rightarrow \alpha = 4$

$\therefore$ Equation of circle is,
$(x - 4)^2 + y^2 = 17$
$\Rightarrow x^2 + y^2 - 8x - 1 = 0$

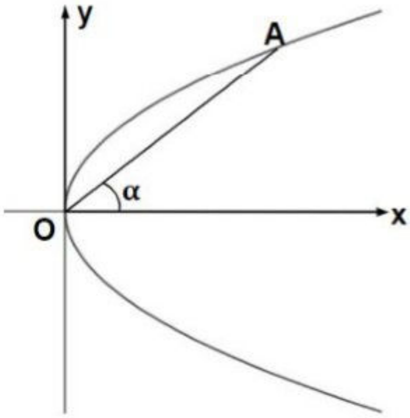
39 The length of the chord of the parabola $y^2 = 4ax$ $(a > 0)$ which passes through the vertex and makes an acute angle $\alpha$ with the axis of the parabola is

A. $\pm 4a \cot\alpha \operatorname{cosec}\alpha$

B. $4a \cot\alpha \operatorname{cosec}\alpha$

C. $-4a \cot\alpha \operatorname{cosec}\alpha$

D. $4a \operatorname{cosec}^2\alpha$




Equation $AO$ is $y - 0 = \tan\alpha(x - 0)$

$\Rightarrow y = x\tan\alpha$

$\because y^2 = 4ax$

$\Rightarrow x^2\tan^2\alpha = 4ax$

$\Rightarrow x\tan^2\alpha = 4a$

$\Rightarrow x = 4a\cot^2\alpha$

$\therefore y = 4a\cot^2\alpha\tan\alpha$

$\Rightarrow y = 4a\cot\alpha$

$\therefore A = (4a\cot^2\alpha, 4a\cot\alpha)$

$$\Rightarrow AO = \sqrt{16a^2\cot^4\alpha + 16a^2\cot^2\alpha}$$
$$= 4a\cot\alpha\sqrt{\cot^2\alpha + 1}$$
$$= 4a\cot\alpha \operatorname{cosec}\alpha$$

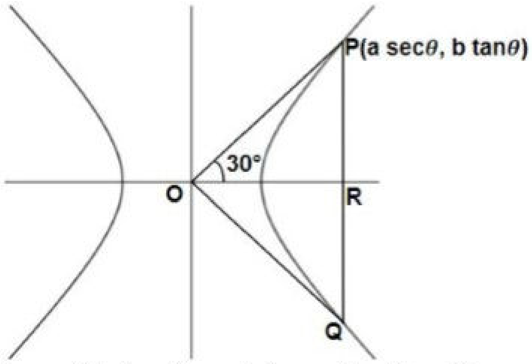
40 A double ordinate $PQ$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is such that $\Delta OPQ$ is equilateral, $O$ being the centre of the hyperbola. Then the eccentricity $e$ satisfies the relation

A. $1 < e < \frac{2}{\sqrt{3}}$

B. $e = \frac{2}{\sqrt{3}}$

C. $e = \frac{\sqrt{3}}{2}$

D. $e > \frac{2}{\sqrt{3}}$




Let any double ordinate $P, Q$ be $(a\sec\theta, b\tan\theta)$ and $(a\sec\theta, -b\tan\theta)$.

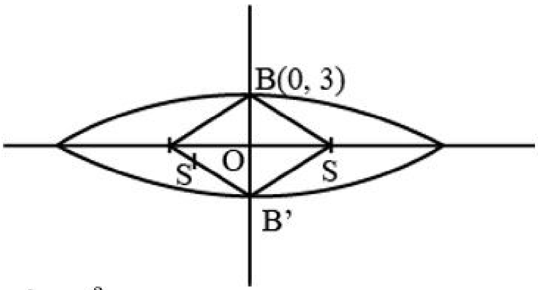
In $\Delta OPR$,

$$\tan 30^\circ = \frac{b\tan\theta}{a\sec\theta}$$
$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{b}{a}\sin\theta$$
$$\Rightarrow \frac{b}{a} = \frac{1}{\sqrt{3}\sin\theta}$$

$$e^2 = 1 + \frac{b^2}{a^2}$$
$$= 1 + \frac{1}{3\sin^2\theta}$$
$$\Rightarrow e^2 > 1 + \frac{1}{3} \ (\because \max[\sin^2\theta] = 1)$$
$$\Rightarrow e^2 > \frac{4}{3}$$
$$\Rightarrow e > \frac{2}{\sqrt{3}}$$

41 If $B$ and $B'$ are the ends of the minor axis and $S$ and $S'$ are foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, then the area of the rhombus $SBS'B'$ will be

- A. $12$ sq. unit
- B. $48$ sq. unit
- C. $24$ sq. unit
- D. $36$ sq. unit




$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

$$e=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$$

$$S=(ae,0)=\left(5\times\frac{4}{5},0\right)=(4,0)$$

Area of Rhombus $=4$ area $\Delta BOS$

$$=4\left(\frac{1}{2}\times OS\times OB\right)$$

$$=4\left(\frac{1}{2}\times 4\times 3\right)$$

$$=24 \text{ sq. units}$$

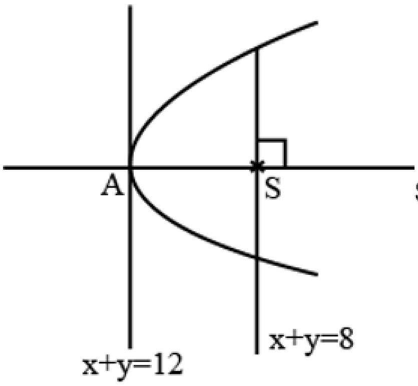
42 The equation of the latus rectum of a parabola is $x+y=8$ and the equation of the tangent at the vertex is $x+y=12$. Then the length of the latus rectum is

A. $4\sqrt{2}$ units

B. $2\sqrt{2}$ units

C. 8 units

D. $8\sqrt{2}$ units




Since, $x+y=8$ and $x+y=12$ are parallel.

Therefore, distance between them is,

$$AS=\left|\frac{C_1-C_2}{\sqrt{a^2+b^2}}\right|=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

Hence, length of the latus rectum is,

$$LR=4AS$$
$$=4\times 2\sqrt{2}$$
$$=8\sqrt{2}\text{ units}$$

43 The equation of the plane through the point $(2, -1, -3)$ and parallel to the lines $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-4}$ and $\frac{x}{2} = \frac{y-1}{-3} = \frac{z-2}{2}$ is

A. $8x + 14y + 13z + 37 = 0$

B. $8x - 14y - 13z - 37 = 0$

C. $8x - 14y - 13z + 37 = 0$

D. $8x - 14y + 13z + 37 = 0$

E. $^{*}x + 2y + 2z + 6 = 0$

Equation of a plane is $a(x-2) + b(y+1) + c(z+3) = 0$
Parallel to lines
$\Rightarrow 2a + 3b - 4c = 0$
$\underline{2a - 3b + 2c = 0}$
$4a = 2c \Rightarrow a : b = 1 : 2$
$\therefore a : b : c = 1 : 2 : 2$
Equation of plane is :
$x - 2 + 2y + 2 + 2z + 6 = 0$
$x + 2y + 2z + 6 = 0$

Note: No option correct in the paper.

44 The sine of the angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is

A. $\frac{2\sqrt{3}}{5}$

B. $\frac{\sqrt{2}}{10}$

C. $\frac{4}{5\sqrt{2}}$

D. $\frac{\sqrt{5}}{6}$

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$
$$\Rightarrow \vec{b}=3\hat{i}+4\hat{j}+5\hat{k}$$

$$2x-2y+z=5$$
$$\Rightarrow \vec{n}=2\hat{i}-2\hat{j}+\hat{k}$$

$$\sin\theta=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}\right|$$
$$=\left|\frac{3\cdot2-4\cdot2+5\cdot1}{\sqrt{9+16+25}\cdot\sqrt{4+4+1}}\right|$$
$$=\frac{3}{5\sqrt{2}\times3}$$
$$=\frac{\sqrt{2}}{10}$$

45 Let $f(x)=\sin\ x+\cos\ ax$ be periodic function. Then

A. $'a'$ is real number

B. $'a'$ is any irrational number

C. $'a'$ is rational number

D. $a=0$

Period of $\sin\ x+\cos\ ax$ is LCM of $1$ and $a$.
But for LCM to exist, here $'a'$ must be a rational number.

46 The domain of $f(x)=\sqrt{\left(\frac{1}{\sqrt{x}}-\sqrt{(x+1)}\right)}$ is

A. $x>-1$

B. $(-1,\infty)\setminus\{0\}$

C. $\left(0,\frac{\sqrt{5}-1}{2}\right]$

D. $\left[\frac{1-\sqrt{5}}{2},0\right)$

$f(x)=\sqrt{\frac{1}{\sqrt{x}}-\sqrt{x+1}}$

$x>0,\ x+1\geq 0\Rightarrow x\geq -1$

$\frac{1}{\sqrt{x}}-\sqrt{x+1}\geq 0\Rightarrow \frac{1}{x}\geq x+1$

$\Rightarrow x^2+x-1\leq 0$

$\Rightarrow x=\frac{-1+\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}$

$\therefore x\in\left(0,\frac{\sqrt{5}-1}{2}\right]$

47 Let $y=f(x)=2x^2-3x+2$. The differential of $y$ when $x$ changes from $2$ to $1.99$ is

A. $0.01$

B. $0.18$

C. $-0.05$

D. $0.07$

$\delta y=f'(x).\,\delta x,\ \text{changes from 2 to 1.99}$

$\Rightarrow\ \delta x=-0.01$

$\Rightarrow \delta y=(4x-3)\delta x$

$\Rightarrow \delta y=(4\times 2-3)(-0.01)=-0.05$

48 If $\lim_{x\to 0}\left[\frac{1+cx}{1-cx}\right]^{1/x}=4$, then $\lim_{x\to 0}\left[\frac{1+2cx}{1-2cx}\right]^{1/x}$ is

A. $2$

B. $4$

C. $16$

D. $64$

Given $\lim_{x\to 0}\left[\frac{1+cx}{1-cx}\right]^{1/x}=4$

$\Rightarrow e^{\lim_{x\to 0}\frac{1}{x}\left[\frac{1+cx-1+cx}{1-cx}\right]}=4$

$\therefore c=\log_e 2$

now $\lim_{x\to 0}\left[\frac{1+2cx}{1-2cx}\right]^{1/x}$

$=e^{\lim_{x\to 0}\frac{1}{x}\left[\frac{1+2cx-1+2cx}{1-2cx}\right]}$

$=e^{4\log_e 2}=16$

49 Let $f: R\to R$ be twice continuously differentiable (or $f''$ exists and is continuous) such that $f(0)=f(1)=f'(0)=0$. Then

A. $f''(c)=0$ for some $c\in R$

B. there is no point for which $f''(x)=0$

C. at all points $f''(x)>0$

D. at all points $f''(x)<0$

Consider $f(x)$ on $[0,1]$
Applying Rolle's theorem on the interval $[0,1]$,
$f'(a)=0$ for some $a\in(0,1)$

Now, applying Rolle's theorem to $f'(x)$on the interval $[0,a]$,
$f''(c)=0$ for some $c\in(0,a)$

50 Let $f(x)=x^{13}+x^{11}+x^9+x^7+x^5+x^3+x+12$. Then

A. $f(x)$ has 13 non-zero real roots

B. $f(x)$ has exactly one real root

C. $f(x)$ has exactly one pair of imaginary roots

D. $f(x)$ has no real root

$f(x)=x^{13}+x^{11}+x^9+x^7+x^5+x^3+x+12$
$f'(x)=13x^{12}+11x^{10}+9x^8+7x^6+5x^4+3x^2+1>0\ \forall\, x\in R$
i.e Monotonically increasing $\forall\, x\in R$.
$\Rightarrow f(x)$ intersects $x-$axis at only one point
$\therefore$ exactly one solution.

51 Let $z_1$ and $z_2$ be two imaginary roots of $z^2+pz+q=0$, where $p$ and $q$ are real. The points $z_1,\ z_2$ and origin form an equilateral triangle if

A. $p^2>3q$

B. $p^2<3q$

C. $p^2=3q$

D. $p^2=q$

Since $p,q\in R$,
$\therefore z_1,z_2$ are in conjugate pairs.

Let $z_1=\alpha+i\beta$ and $z_2=\alpha-i\beta$

$z_1 + z_2 = 2\alpha = -p$
$z_1 z_2 = \alpha^2 + \beta^2 = q$

Solving the two equations, we get

$z_1 \equiv A\left(\frac{-p}{2}, \sqrt{\frac{4q-p^2}{4}}\right)$

$z_2 \equiv B\left(\frac{-p}{2}, -\sqrt{\frac{4q-p^2}{4}}\right)$

$O \equiv (0, 0)$

$z_1,\ z_2$ and origin form an equilateral triangle.
So, $OA = AB$

$\Rightarrow \frac{p^2}{4} + \frac{4q-p^2}{4} = \left(\sqrt{\frac{4q-p^2}{4}} + \sqrt{\frac{4q-p^2}{4}}\right)^2$

$\Rightarrow q = 4q - p^2$
$\Rightarrow p^2 = 3q$

52 If the vectors $\vec{\alpha} = \hat{i} + a\hat{j} + a^2\hat{k}$, $\vec{\beta} = \hat{i} + b\hat{j} + b^2\hat{k}$ and $\vec{\gamma} = \hat{i} + c\hat{j} + c^2\hat{k}$ are three non-coplanar vectors and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, then the value of $abc$ is

A. $1$

B. $0$

C. $-1$

D. $2$

$$\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\because \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

$$\therefore\ abc + 1 = 0$$

53 If the line $y=x$ is a tangent to the parabola $y=ax^2+bx+c$ at the point $(1,1)$ and the curve passes through $(-1,0)$, then

A. $a=b=-1,\ c=3$

B. $a=b=\frac{1}{2},\ c=0$

C. $a=c=\frac{1}{4},\ b=\frac{1}{2}$

D. $a=0,\ b=c=\frac{1}{2}$

$y=x$ is a tangent
$\therefore$ slopes are equal.
$\frac{dy}{dx}=2ax+b$
$\Rightarrow 1=2a+b$ at $(1,1)\cdots(1)$
Also, the parabola passes through $(1,1)$
$\Rightarrow a+b+c=1\cdots(2)$
The parabola passes through $(-1,0)$
$\Rightarrow 0=a-b+c\cdots(3)$
Solving $(1),(2),(3)$, we get -
$\therefore a=c=\frac{1}{4}$
and $b=\frac{1}{2}$

54 In an open interval $\left(0,\frac{\pi}{2}\right)$,

A. $\cos x+x\sin x<1$

B. $\cos x+x\sin x>1$

C. no specific order relation can be ascertained between $\cos x+x\sin x$ and $1$

D. $\cos x+x\sin x<\frac{1}{2}$

Let $\cos x+x\sin x=f(x)$
$f'(x)=-\sin x+\sin x+x\cos x$
$\Rightarrow f'(x)=x\cos x>0\ \forall\ x\ \in\left(0,\frac{\pi}{2}\right)$
$\Rightarrow x>0$
$\Rightarrow f(x)>f(0)$
$\Rightarrow \cos x+x\sin x>1$

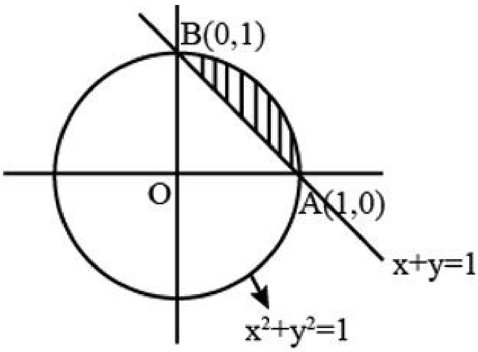
55 The area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$ is

A. $\frac{\pi^2}{2}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{4} - \frac{1}{2}$

D. $\frac{\pi^2}{3}$




Required area(shaded)

$= \frac{1}{4}(\text{area of circle}) - \text{area } (\Delta AOB)$

$= \frac{1}{4} \cdot \pi(1)^2 - \frac{1}{2} \cdot 1 \cdot 1$

$= \frac{\pi}{4} - \frac{1}{2} \text{sq.units.}$

56 If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$ [a, b, c, d are all real], then P(x).Q(x) = 0 has

A. at least two real roots

B. two real roots

C. four real roots

D. no real root

$P(x) = ax^2 + bx + c$
$Q(x) = -ax^2 + dx + c$
$D_1 = b^2 - 4ac$
$D_2 = d^2 + 4ac$

If $ac$ is negative, then $D_1$ will have 2 real roots ($D_2$ too can have real roots)

If $ac$ is positive,then $D_2$ will have 2 real roots ($D_1$ too can have real roots)



57 Let $A = \{x \epsilon R : -1 \leq x \leq 1\}$ & $f : A \rightarrow A$ be a mapping defined by $f(x) = x|x|$. Then $f$ is

A. injective but not surjective

B. surjective but not injective

C. neither injective nor surjective

D. bijective

$f(x) = x|x|$

$$f(x) = \left\{ \begin{array}{ll} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{array} \right\}$$

$\therefore$ it is one-one and onto

58 Let $f(x)=\sqrt{x^2-3x+2}$ and $g(x)=\sqrt{x}$ be two given functions . If $S$ be the domain of $f\circ g$ and $T$ be the domain of $g\circ f$, then

A. $S=T$

B. $S\cap T=\varphi$

C. $S\cap T$ is a singleton

D. $S\cap T$ is an interval

$f(x)=\sqrt{x^2-3x+2},\ g(x)=\sqrt{x}$

$f\circ g(x)=f(\sqrt{x})=\sqrt{x-3\sqrt{x}+2},$

$x-3\sqrt{x}+2\geq 0,\ x>0$

$\Rightarrow x+2\geq 3\sqrt{x}$

$\Rightarrow x^2+4+4x-9x\geq 0$

$\Rightarrow x^2-5x+4\geq 0$

$(x-1)(x-4)\geq 0$

$S=x\in(0,1]\cup[4,\infty)$

$g\circ f(x)=g(\sqrt{x^2-3x+2})$

$=\sqrt{\sqrt{x^2-3x+2}}\Rightarrow x^2-3x+2\geq 0$

$\Rightarrow(x-1)(x-2)\geq 0$

$T\Rightarrow x\in(-\infty,1]\cup[2,\infty)$

$\therefore\ S\cap T=(0,1]\cup[4,\infty)$

59 Let $\rho_1$ and $\rho_2$ be two equivalence relations defined on a non-void set S. Then

- A. both $\rho_1 \cap \rho_2$ and $\rho_1 \cup \rho_2$ are equivalence relations.
- B. $\rho_1 \cap \rho_2$ is equivalence relation but $\rho_1 \cup \rho_2$ is not so.
- C. $\rho_1 \cup \rho_2$ is equivalence relation but $\rho_1 \cap \rho_2$ is not so.
- D. neither $\rho_1 \cap \rho_2$ nor $\rho_1 \cup \rho_2$ is equivalence relation

Given $\rho_1, \rho_2$ are equivalence relations on $S$.
$\Rightarrow \rho_1, \rho_2$ are reflexive, symmetric and transitive.

Reflexive:
Let $x \in S$
$\Rightarrow (x, x) \in \rho_1$ and $(x, x) \in \rho_2$
$\Rightarrow (x, x) \in \rho_1 \cap \rho_2$
$\Rightarrow \rho_1 \cap \rho_2$ is reflexive.

Symmetric:
Let $(x, y) \in \rho_1 \cap \rho_2$
We have to show $(y, x) \in \rho_1 \cap \rho_2$
$(x, y) \in \rho_1 \cap \rho_2$
$\Rightarrow (x, y) \in \rho_1$ and $(x, y) \in \rho_2$
$\Rightarrow (y, x) \in \rho_1$ and $(y, x) \in \rho_2$
$\Rightarrow (y, x) \in \rho_1 \cap \rho_2$
$\Rightarrow \rho_1 \cap \rho_2$ is symmetric.

Transitive:
Let $(x, y), (y, z) \in \rho_1 \cap \rho_2$
$\Rightarrow (x, y), (y, z) \in \rho_1$ and $(x, y), (y, z) \in \rho_2$
$\Rightarrow (x, z) \in \rho_1$ and $(x, z) \in \rho_2$
$\Rightarrow (x, z) \in \rho_1 \cap \rho_2$
$\Rightarrow \rho_1 \cap \rho_2$ is transitive.

Therefore, $\rho_1 \cap \rho_2$ is equivalence relation.

$\rho_1 \cup \rho_2$ is always reflexive and symmetric but not transitive.
e.g. Let $S = \{1, 2, 3\}$
$\rho_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$
$\rho_2 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

$\rho_1, \rho_2$ is equivalence relation.
But $\rho_1 \cup \rho_2$ is not transitive as $(1,2), (2,3) \in \rho_1 \cup \rho_2$ but $(1,3) \notin \rho_1 \cup \rho_2$

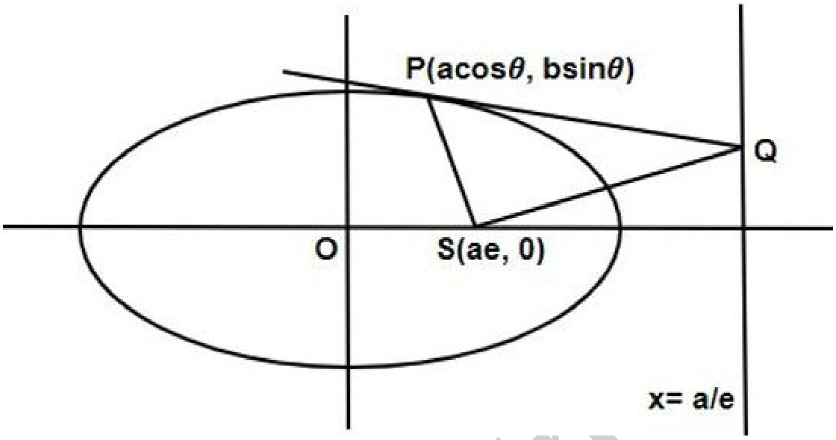
60 Consider the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The portion of the tangent at any point of the curve intercepted between the point of contact and the directrix subtends at the corresponding focus an angle of

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{\pi}{6}$



Equation of tangent at any point $P(a\cos\theta, b\sin\theta)$ on the ellipse is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

When $x=\frac{a}{e}, y=\left(1-\frac{\cos\theta}{e}\right)\frac{b}{\sin\theta}$

$$\therefore Q\equiv\left(\frac{a}{e},\left(1-\frac{\cos\theta}{e}\right)\frac{b}{\sin\theta}\right)$$

Now, Slope of $SQ\times$ Slope of $PS$

$$=\frac{\left(1-\frac{\cos\theta}{e}\right)\frac{b}{\sin\theta}}{\frac{a}{e}-ae}\times\frac{b\sin\theta}{a\cos\theta-ae}$$

$$=\frac{(e-\cos\theta)b}{a(1-e^2)\sin\theta}\times\frac{b\sin\theta}{a(\cos\theta-e)}$$

$$=\frac{-b^2}{a^2(1-e^2)}$$
$$=\frac{-b^2\cdot a^2}{a^2\cdot b^2}$$
$$=-1$$

So, the angle between the line $PS$ and $SQ$ is $\frac{\pi}{2}$.

61 A line cuts $x$-axis at $A(7,0)$ and the $y$-axis at $B(0,-5)$. A variable line $PQ$ is drawn perpendicular to $AB$ cutting the $x$-axis at $P(a,0)$ and the $y$-axis at $Q(0,b)$. If $AQ$ and $BP$ intersect at $R$, the locus of $R$ is

A. $x^2+y^2+7x+5y=0$

B. $x^2+y^2+7x-5y=0$

C. $x^2+y^2-7x+5y=0$

D. $x^2+y^2-7x-5y=0$

Slope of the line $AB$ is $\frac{5}{7}$

Slope of the line $PQ$

$\frac{-b}{a}=\frac{-7}{5}$

$\frac{b}{a}=\frac{7}{5}$

Equation of the line

$BP: \frac{x}{a}-\frac{y}{5}=1$

$\frac{1}{a}=\frac{5+y}{5x}\cdots(1)$

Equation of the line

$AQ: \frac{x}{7}+\frac{y}{b}=1$

$\frac{1}{b}=\frac{7-x}{7y}\cdots(2)$

solving above equations we get

$x^2+y^2-7x+5y=0$

62 Let $0 < \alpha < \beta < 1$. Then $\lim_{n\to\infty} \sum_{k=1}^{n} \int_{1/(k+\beta)}^{1/(k+\alpha)} \frac{dx}{1+x}$ is

A. $\log_e \frac{\beta}{\alpha}$

B. $\log_e \frac{1+\beta}{1+\alpha}$

C. $\log_e \frac{1+\alpha}{1+\beta}$

D. $\infty$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{1/(k+\beta)}^{1/(k+\alpha)} \frac{dx}{1+x}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\ln(1+x)\right]_{1/(k+\beta)}^{1/(k+\alpha)}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\ln\left(1+\frac{1}{k+\alpha}\right) - \ln\left(1+\frac{1}{k+\beta}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[\left(\frac{k+\beta}{k+\alpha}\right)\left(\frac{k+\alpha+1}{k+\beta+1}\right)\right]$$

$$= \ln\left[\frac{\beta+1}{\alpha+1} \times \frac{\alpha+2}{\beta+2} \times \frac{\beta+2}{\alpha+2} \times \frac{\alpha+3}{\beta+3} \times \ldots\right]$$

$$= \log_e \frac{1+\beta}{1+\alpha}$$

63 $\lim_{x\to 1}\left[\frac{1}{\ln x}-\frac{1}{(x-1)}\right]$

A. Does not exist

B. 1

C. $\frac{1}{2}$

D. 0

$$\lim_{x\to 1}\left(\frac{1}{\log x}-\frac{1}{(x-1)}\right)$$

$$\Rightarrow \lim_{x\to 1}\left(\frac{x-1-\log x}{(x-1)\log x}\right)$$

Using L'Hospital's Rule, we get

$$\Rightarrow \lim_{x\to 1}\left(\frac{1-0-\frac{1}{x}}{\log x+\frac{x-1}{x}}\right)$$

Using L'Hospital's Rule again, we get

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}\right\}=\frac{1}{2}$$

64 Let $y = \frac{1}{1+x+\ln\ x}$, Then

A. $x\frac{dy}{dx} + y = x$

B. $x\frac{dy}{dx} = y(y\ \ln\ x - 1)$

C. $x^2\frac{dy}{dx} = y^2 + 1 - x^2$

D. $x\left[\frac{dy}{dx}\right]^2 = y - x$

$$y = \frac{1}{1+x+\ln\ x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{(1+x+\ln\ x)^2}\left(1+\frac{1}{x}\right)$$

$$\Rightarrow x\frac{dy}{dx} = -y^2\left(\frac{1}{y} - \ln x\right)$$

$$\Rightarrow x\frac{dy}{dx} = y(y\ \ln\ x - 1)$$

65 Consider the curve $y = be^{-x/a}$ where $a$ and $b$ are non-zero real numbers. Then

A. $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve at $(0, 0)$.

B. $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve where the curve crosses the axis of $y$.

C. $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve at $(a, 0)$

D. $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve at $(2a, 0)$.

$y = be^{-x/a}$

$\Rightarrow \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$

Slope of the tangent $\frac{x}{a} + \frac{y}{b} = 1$ is $\frac{-b}{a}$

Also, $\frac{dy}{dx} = \frac{-b}{a}$ when $x = 0$

So, $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve at the point where $x = 0$

$x = 0 \Rightarrow y = b$

So, $\frac{x}{a} + \frac{y}{b} = 1$ is tangent to the curve at the point $(0, b)$

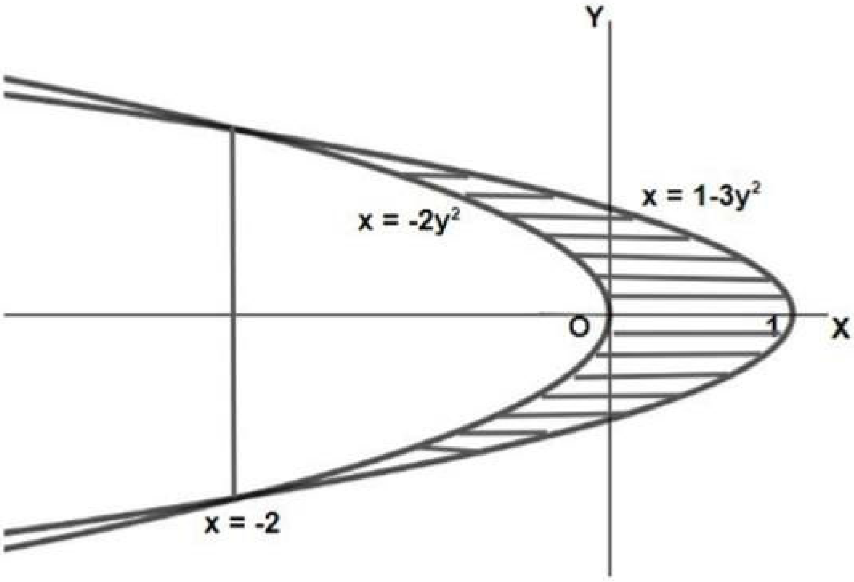
66 The area of the figure bounded by the parabola $x = -2y^2, x = 1 - 3y^2$ is

A. $\frac{1}{3}$ square unit

B. $\frac{4}{3}$ square unit

C. 1 square unit

D. 2 square unit




Required area

$$= 2\int_0^1 (1 - 3y^2 - (-2y^2))dy$$

$$= 2\int_0^1 (1 - y^2)dy$$

$$= 2\left[y - \frac{y^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

67 A particle is projected vertically upwards. If it has to stay above the ground for 12 seconds, then

✓ A velocity of projection is $192 \text{ ft/sec}$

B greatest height attained is $600 \text{ ft}$

C velocity of projection is $196 \text{ ft/sec}$

✓ D greatest height attained is $576 \text{ ft}$

$v = u - gt$
$v = 0,\ g = 32 \text{ ft/sec},\ t = 6 \text{ sec}$
$\Rightarrow 0 = u - 32 \times 6$
$\Rightarrow u = 192 \text{ ft/sec}$

$h = ut - \frac{1}{2}gt^2$
$\Rightarrow h = 192 \times 6 - \frac{1}{2} \times 32 \times 6^2$
$= 576 \text{ ft}$

68 The equation $x^{(\log_3 x)^2 - \frac{9}{2}\log_3 x + 5} = 3\sqrt{3}$ has

✓ A at least one real root

B exactly one real root

✓ C exactly one irrational root

D complex roots

Let $\log_3 x = t \Rightarrow x = 3^t$
Then $(3^t)^{t^2 - \frac{9}{2}t + 5} = 3\sqrt{3}$
$\Rightarrow 3^{t^3 - \frac{9}{2}t^2 + 5t} = 3^{\frac{3}{2}}$
$\Rightarrow t^3 - \frac{9}{2}t^2 + 5t = \frac{3}{2}$
$\Rightarrow 2t^3 - 9t^2 + 10t - 3 = 0$
$\Rightarrow \left(t - \frac{1}{2}\right)(t-1)(t-3) = 0$
$\Rightarrow t = \frac{1}{2}, 1, 3$

$\therefore x = \sqrt{3}, 3, 27$ are the roots of the given equation.

69 In a certain test, there are $n$ questions. In this test $2^{n-i}$ students gave wrong answers to at least $i$ questions, where $i = 1, 2....., n$. If the total number of wrong answers given is $2047$, then $n$ is equal to ,

A. $10$

B. $11$

C. $12$

D. $13$

$2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 2^0 = 2047$

$\Rightarrow \dfrac{1(2^n - 1)}{2 - 1} = 2047$

$\Rightarrow 2^n - 2048$

$\Rightarrow 2^n = 2^{11}$

$\Rightarrow n = 11$

70 $A$ and $B$ are independent events. The probability that both $A$ and $B$ occur is $\frac{1}{20}$ and the probability that neither of them occurs is $\frac{3}{5}$. The probability of occurence of $A$ is

A $\frac{1}{2}$

B $\frac{1}{10}$

✓ C $\frac{1}{4}$

✓ D $\frac{1}{5}$

$P(A \cap B) = P(A)P(B) = \frac{1}{20} \Rightarrow P(B) = \frac{1}{20P(A)}$

$P(\bar{A} \cap \bar{B}) = \frac{3}{5} = 1 - P(A \cup B)$

$\Rightarrow \frac{3}{5} = 1 - P(A) - P(B) + P(A \cap B)$

$\Rightarrow \frac{3}{5} = 1 - P(A) - \frac{1}{20P(A)} + \frac{1}{20}$

$\Rightarrow \frac{3}{5} = \frac{21}{20} - P(A) - \frac{1}{20P(A)}$

$\Rightarrow \frac{12-21}{20} = -P(A) - \frac{1}{20P(A)}$

Let $P(A) = x$

$\Rightarrow \frac{-9}{20} = \frac{-20x^2 - 1}{20x}$

$\Rightarrow 20x^2 - 9x + 1 = 0$

$\Rightarrow (4x-1)(5x-1) = 0$

$\Rightarrow x = \frac{1}{4}, \frac{1}{5}$

$i.e.\ P(A) = \frac{1}{4},\ P(A) = \frac{1}{5}$

71 The equation of the straight line passing through the point $(4,3)$ and making intercepts on the co-ordinate axes whose sum is $-1$ is

✓ A $\frac{x}{2}-\frac{y}{3}=1$

✓ B $\frac{x}{-2}+\frac{y}{1}=1$

C $-\frac{x}{3}+\frac{y}{2}=1$

D $\frac{x}{1}-\frac{y}{2}=1$

$\frac{x}{a}+\frac{y}{b}=1 \rightarrow$ straight line

passes through $(4,3)=1$

$\Rightarrow \frac{4}{a}+\frac{3}{b}=1$

$\text{sum}=-1 \Rightarrow a+b=-1$

$\Rightarrow \frac{4}{a}+\frac{3}{-1-a}=1$

$-4-4a+3a=-a-a^2$

$-4-\not{a}=-\not{a}-a^2 \Rightarrow a=\pm 2$

$a=2, b=-3 \quad \& \quad a=-2, b=1$

$\therefore \text{Lines are } \frac{x}{2}-\frac{y}{3}=1, -\frac{x}{2}+y=1$

72 Let $f(x) = \frac{1}{3}x \sin x - (1 - \cos x)$. The smallest positive integer $k$ such that $\lim_{x \to 0} \frac{f(x)}{x^k} \neq 0$ is

A. 4

B. 3

C. 2 ✓

D. 1

$$\lim_{x \to 0} \frac{f(x)}{x^k} = \lim_{x \to 0} \frac{\frac{x \sin x}{3} - 1 + \cos x}{x^k} \quad [\frac{0}{0} \text{form}]$$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{3} + \frac{x \cos x}{3} - \sin x}{kx^{k-1}} \quad \left[\frac{0}{0} \text{form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{\cos x}{3} + \frac{\cos x}{3} - \frac{x \sin x}{3} - \cos x}{k(k-1)x^{k-2}} \neq 0 \text{ if } k = 2$$

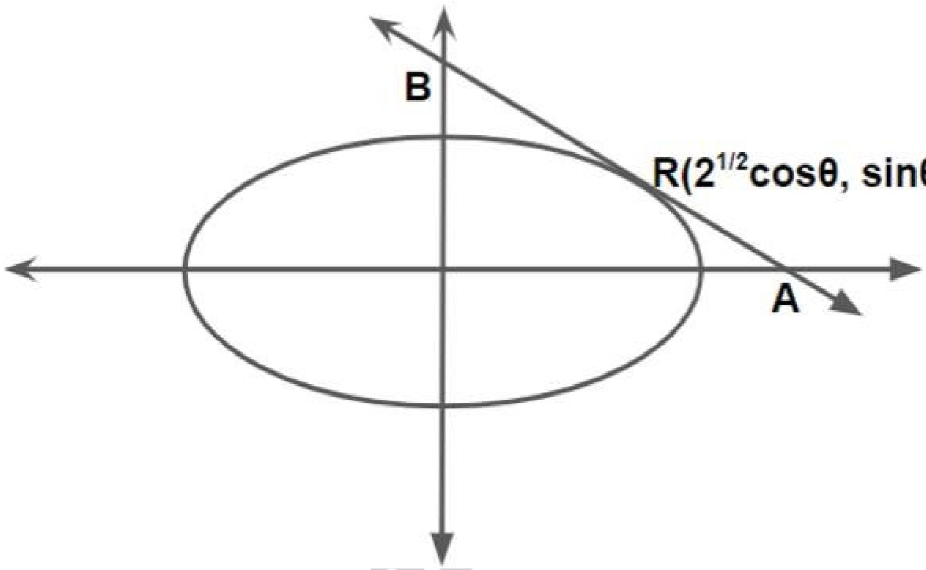
73 Consider a tangent to the ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$ at any point. The locus of the midpoint of the portion intercepted between the axes is

A. $\frac{x^2}{2}+\frac{y^2}{4}=1$

B. $\frac{x^2}{4}+\frac{y^2}{2}=1$

C. $\frac{1}{3x^2}+\frac{1}{4y^2}=1$

D. $\frac{1}{2x^2}+\frac{1}{4y^2}=1$



Tangent at $R\,(\sqrt{2}\sin\theta, \sin\theta)$

$$\Rightarrow \frac{x\sqrt{2}\ \cos\theta}{2} + \frac{y\ \sin\theta}{1} = 1$$

$$A = \left(\frac{\sqrt{2}}{\cos\theta}, 0\right),\quad B = \left(0, \frac{1}{\sin\theta}\right)$$

Let $p(h, k)$ be the locus of the midpoint.

$$\therefore (h, k) = \left(\frac{\sqrt{2}}{2\cos\theta}, \frac{1}{2\sin\theta}\right)$$

$$\therefore h = \frac{1}{\sqrt{2}\ \cos\theta},\ k = \frac{1}{2\sin\theta}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}\ h},\ \sin\theta = \frac{1}{2k}$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = \frac{1}{2h^2} + \frac{1}{4k^2}$$

$$\therefore \text{ Locus of } (h, k) \text{ is } \frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

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74 Tangent is drawn at any point $P(x, y)$ on a curve, which passes through $(1, 1)$. The tangent cuts $X$-axis and $Y$-axis at $A$ and $B$ respectively. If $AP : BP = 3 : 1$, then

✓ A the differential equation of the curve is $3x\frac{dy}{dx} + y = 0$

B the differential equation of the curve is $3x\frac{dy}{dx} - y = 0$

✓ C the curve passes through $\left(\frac{1}{8}, 2\right)$

D the normal at $(1, 1)$ is $x + 3y = 4$

Since $BP : AP = 3 : 1$

Equation of the tangent is $Y - y = f'(x)(X - x)$.

Intercept on $X$-axis $\left(x - \frac{y}{f'(x)}, 0\right)$, Y-axis$(0, y - xf'(x))$.

Now $x = \frac{\left(x - \frac{y}{f'(x)}\right) + 1 \times 0}{3 + 1}$ {since, it divides internally $3 : 1$}

$\Rightarrow \frac{dy}{dx} = -\frac{y}{3x} \Rightarrow \frac{dy}{y} = -\frac{dx}{3x} \Rightarrow 3x\frac{dy}{dx} + y = 0$

$\Rightarrow \log\, y = -\frac{1}{3}\log\, x + \log\, C \Rightarrow xy^3 = C$

But curve passes through $(1, 1) \Rightarrow 1 = C$

$\therefore\ xy^3 = 1$

$\therefore$ curve passes through $\left(\frac{1}{8}, 2\right)$.

75 Let $y = \frac{x^2}{(x+1)^2(x+2)}$. Then $\frac{d^2y}{dx^2}$ is

A. $2\left[\frac{3}{(x+1)^4} - \frac{3}{(x+1)^3} + \frac{4}{(x+2)^3}\right]$

B. $3\left[\frac{2}{(x+1)^3} + \frac{4}{(x+1)^2} - \frac{5}{(x+2)^3}\right]$

C. $\left[\frac{6}{(x+1)^3} - \frac{4}{(x+1)^2} + \frac{3}{(x+1)^3}\right]$

D. $\left[\frac{7}{(x+1)^3} - \frac{3}{(x+1)^2} + \frac{2}{(x+1)^3}\right]$

$y = \frac{x^2}{(x+1)^2(x+2)}$

$y = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

$\Rightarrow A = -3,\ B = 1,\ C = 4$

$\therefore y = \frac{-3}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{x+2}$

$\frac{dy}{dx} = \frac{3}{(x+1)^2} - \frac{2}{(x+1)^3} - \frac{4}{(x+2)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{6}{(x+1)^3} + \frac{6}{(x+1)^4} + \frac{8}{(x+2)^3}$

$\Rightarrow \frac{d^2y}{dx^2} = 2\left[-\frac{3}{(x+1)^3} + \frac{3}{(x+1)^4} + \frac{4}{(x+2)^3)}\right]$