General Aptitude (GA)

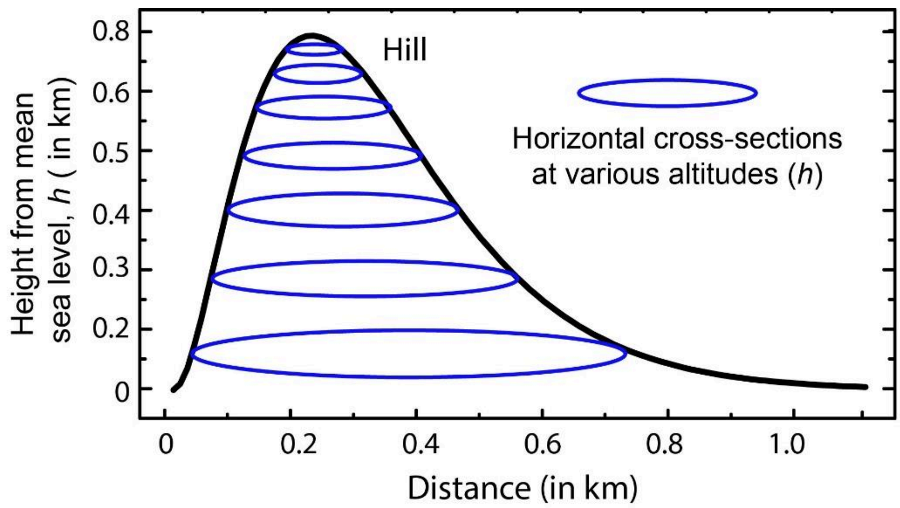
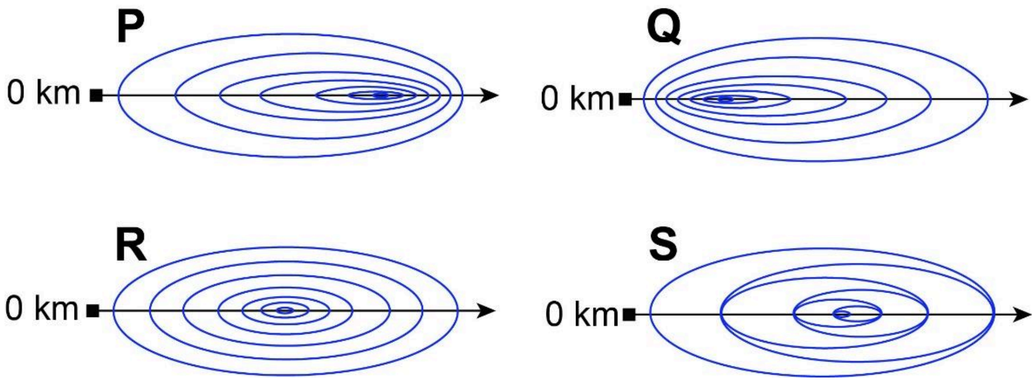
Q.1 – Q.5 Carry ONE mark Each

Q.1	The village was nestled in a green spot, _______ the ocean and the hills.
(A)	through
(B)	in
(C)	at
(D)	between

Q.2	Disagree : Protest : : Agree : _______ (By word meaning)
(A)	Refuse
(B)	Pretext
(C)	Recommend
(D)	Refute

Q.3	A ‘frabjous’ number is defined as a 3 digit number with all digits odd, and no two adjacent digits being the same. For example, 137 is a frabjous number, while 133 is not. How many such frabjous numbers exist?
(A)	125
(B)	720
(C)	60
(D)	80

Q.4	Which one among the following statements must be TRUE about the mean and the median of the scores of all candidates appearing for GATE 2023?
(A)	The median is at least as large as the mean.
(B)	The mean is at least as large as the median.
(C)	At most half the candidates have a score that is larger than the median.
(D)	At most half the candidates have a score that is larger than the mean.

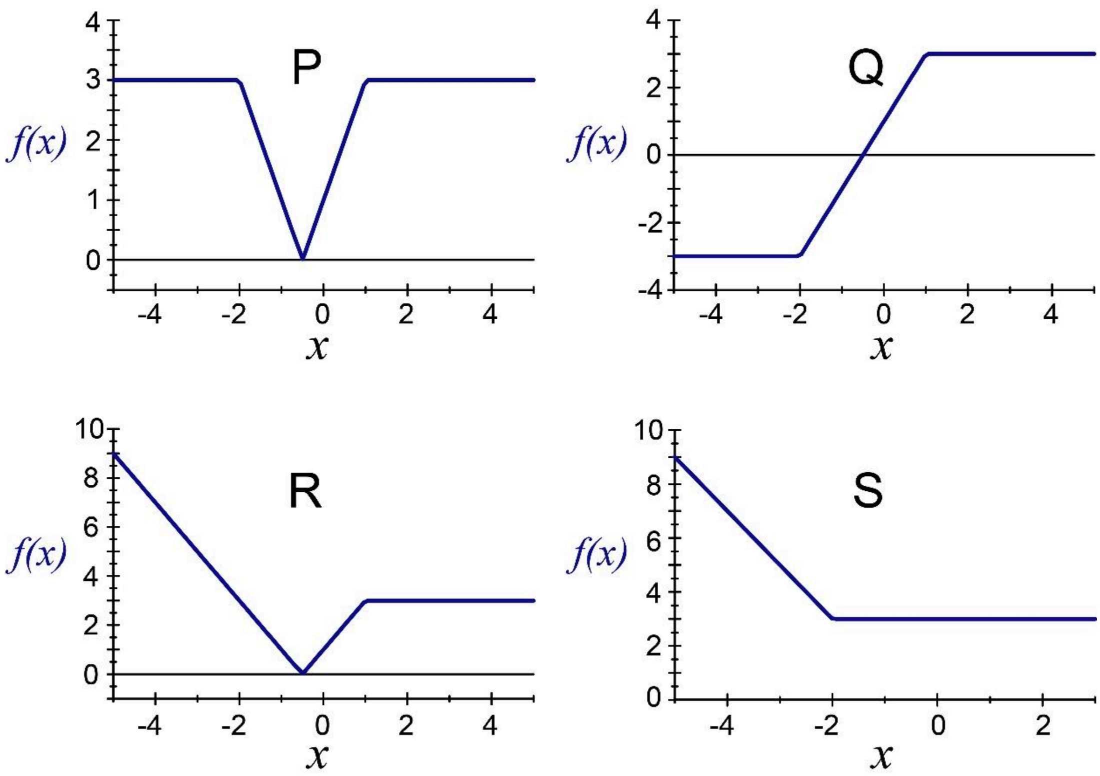
Q.5	In the given diagram, ovals are marked at different heights (h) of a hill. Which one of the following options **P**, **Q**, **R**, and **S** depicts the top view of the hill?
	 
(A)	**P**
(B)	**Q**
(C)	**R**
(D)	**S**

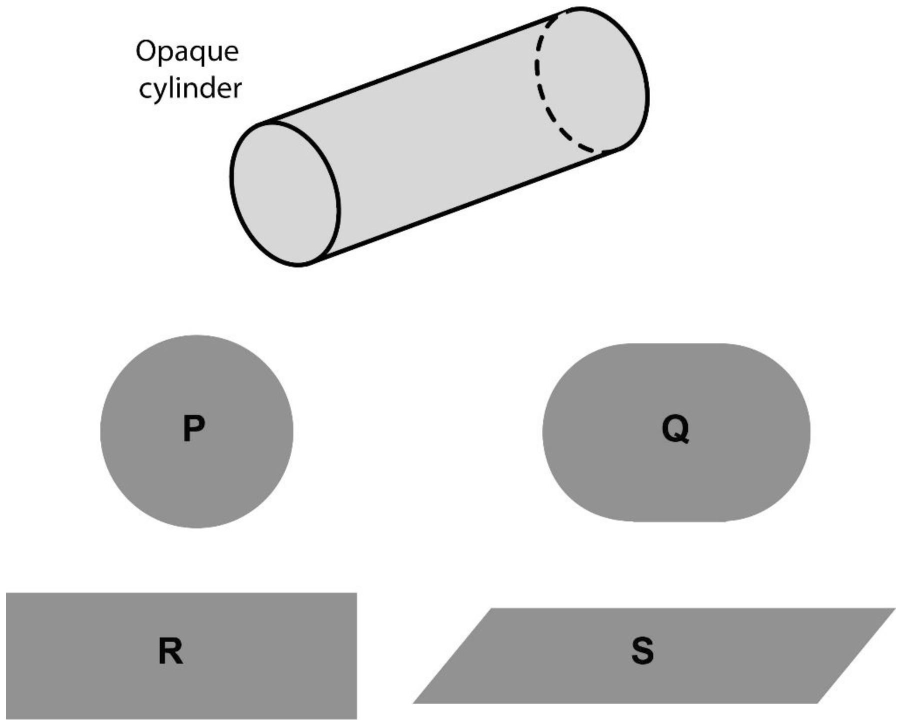
Q.6 – Q.10 Carry TWO marks Each

Q.6	Residency is a famous housing complex with many well-established individuals among its residents. A recent survey conducted among the residents of the complex revealed that all of those residents who are well established in their respective fields happen to be academicians. The survey also revealed that most of these academicians are authors of some best-selling books. Based only on the information provided above, which one of the following statements can be logically inferred with *certainty*?
(A)	Some residents of the complex who are well established in their fields are also authors of some best-selling books.
(B)	All academicians residing in the complex are well established in their fields.
(C)	Some authors of best-selling books are residents of the complex who are well established in their fields.
(D)	Some academicians residing in the complex are well established in their fields.

Q.7	Ankita has to climb 5 stairs starting at the ground, while respecting the following rules: 1. At any stage, Ankita can move either one or two stairs up. 2. At any stage, Ankita cannot move to a lower step. Let $F(N)$ denote the number of possible ways in which Ankita can reach the N^{th} stair. For example, $F(1) = 1,\ F(2) = 2,\ F(3) = 3$. The value of $F(5)$ is _______.
(A)	8
(B)	7
(C)	6
(D)	5

Q.8	The information contained in DNA is used to synthesize proteins that are necessary for the functioning of life. DNA is composed of four nucleotides: Adenine (A), Thymine (T), Cytosine (C), and Guanine (G). The information contained in DNA can then be thought of as a sequence of these four nucleotides: A, T, C, and G. DNA has coding and non-coding regions. Coding regions—where the sequence of these nucleotides are read in groups of three to produce individual amino acids—constitute only about 2% of human DNA. For example, the triplet of nucleotides CCG codes for the amino acid glycine, while the triplet GGA codes for the amino acid proline. Multiple amino acids are then assembled to form a protein. Based only on the information provided above, which of the following statements can be logically inferred with *certainty*? (i) The majority of human DNA has no role in the synthesis of proteins. (ii) The function of about 98% of human DNA is not understood.
(A)	only (i)
(B)	only (ii)
(C)	both (i) and (ii)
(D)	neither (i) nor (ii)

Q.9	Which one of the given figures P, Q, R and S represents the graph of the following function? $$f(x) = \lvert\, \lvert x+2 \rvert - \lvert x-1 \rvert \,\rvert$$
	
(A)	P
(B)	Q
(C)	R
(D)	S

Q.10	An opaque cylinder (shown below) is suspended in the path of a parallel beam of light, such that its shadow is cast on a screen oriented perpendicular to the direction of the light beam. The cylinder can be reoriented in any direction within the light beam. Under these conditions, which one of the shadows **P**, **Q**, **R**, and **S** is NOT possible?
	
(A)	**P**
(B)	**Q**
(C)	**R**
(D)	**S**

USEFUL DATA

$\mathbb{N}$	Set of all positive integers
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i = 1,2, \dots, n\}$
$\mathbb{R} \times \mathbb{R}$	Cartesian product of $\mathbb{R}$ with $\mathbb{R}$

Q.11 – Q.35 Carry ONE mark Each

Q.11	Let $f,\ g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $$f(x,y) = x^2 - \frac{3}{2}xy^2 \text{ and } g(x,y) = 4x^4 - 5x^2y + y^2$$ for all $(x,y) \in \mathbb{R}^2$. Consider the following statements: P: f has a saddle point at $(0,0)$. Q: g has a saddle point at $(0,0)$. Then
(A)	both P and Q are TRUE
(B)	P is FALSE but Q is TRUE
(C)	P is TRUE but Q is FALSE
(D)	both P and Q are FALSE

Q.12	Let $\mathbb{R}^3$ be a topological space with the usual topology and $\mathbb{Q}$ denote the set of rational numbers. Define the subspaces X, Y, Z and W of $\mathbb{R}^3$ as follows: $X = \{(x, y, z) \in \mathbb{R}^3 : \ \lvert x\rvert + \lvert y\rvert + \lvert z\rvert \in \mathbb{Q}\}$ $Y = \{(x, y, z) \in \mathbb{R}^3 : \ xyz = 1\}$ $Z = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\}$ $W = \{(x, y, z) \in \mathbb{R}^3 : \ xyz = 0\}$ Which of the following statements is correct?
(A)	X is homeomorphic to Y
(B)	Z is homeomorphic to W
(C)	Y is homeomorphic to W
(D)	X is NOT homeomorphic to W

Q.13	Let $P(x) = 1 + e^{2\pi ix} + 2\,e^{3\pi ix},\ x \in \mathbb{R},\ i = \sqrt{-1}$. Then $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} P(\,k\sqrt{2}\,)$$ is equal to
(A)	0
(B)	1
(C)	3
(D)	4

Q.14	Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying $T(1,0,0) = (0,1,1),\ T(1,1,0) = (1,0,1)$ and $T(1,1,1) = (1,1,2)$. Then
(A)	T is one-one but T is NOT onto
(B)	T is one-one and onto
(C)	T is NEITHER one-one NOR onto
(D)	T is NOT one-one but T is onto

Q.15	Let $\mathbb{D} = \{z \in \mathbb{C} : \ \|z\| < 1\}$ and $f: \mathbb{D} \to \mathbb{C}$ be defined by $$f(z) = z - 25z^3 + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!}$$ Consider the following statements: P: f has three zeros (counting multiplicity) in $\mathbb{D}$. Q: f has one zero in $\mathbb{U} = \left\{z \in \mathbb{C} : \frac{1}{2} < \|z\| < 1\right\}$. Then
(A)	P is TRUE but Q is FALSE
(B)	P is FALSE but Q is TRUE
(C)	both P and Q are TRUE
(D)	both P and Q are FALSE

Q.16	Let $\mathcal{N} \subseteq \mathbb{R}$ be a non-measurable set with respect to the Lebesgue measure on $\mathbb{R}$. Consider the following statements: P: If $M = \{ x \in \mathcal{N} : x$ is irrational $\}$, then M is Lebesgue measurable. Q: The boundary of $\mathcal{N}$ has positive Lebesgue outer measure. Then
(A)	both P and Q are TRUE
(B)	P is FALSE and Q is TRUE
(C)	P is TRUE and Q is FALSE
(D)	both P and Q are FALSE

Q.17	For $k \in \mathbb{N}$, let E_k be a measurable subset of $[0,1]$ with Lebesgue measure $\frac{1}{k^2}$. Define $$E = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k \text{ and } F = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k$$ Consider the following statements: P: Lebesgue measure of E is equal to zero. Q: Lebesgue measure of F is equal to zero. Then
(A)	both P and Q are TRUE
(B)	both P and Q are FALSE
(C)	P is TRUE but Q is FALSE
(D)	Q is TRUE but P is FALSE

Q.18	Consider $\mathbb{R}^2$ with the usual Euclidean metric. Let $X = \left\{\left(x, x\sin\frac{1}{x}\right) \in \mathbb{R}^2 : x \in (0,1]\right\} \cup \{(0,y) \in \mathbb{R}^2 : -\infty < y < \infty\}$ and $Y = \left\{\left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 : x \in (0,1]\right\} \cup \{(0,y) \in \mathbb{R}^2 : -\infty < y < \infty\}$. Consider the following statements: P: X is a connected subset of $\mathbb{R}^2$. Q: Y is a connected subset of $\mathbb{R}^2$. Then
(A)	both P and Q are TRUE
(B)	P is FALSE and Q is TRUE
(C)	P is TRUE and Q is FALSE
(D)	both P and Q are FALSE

Q.19	Let $M = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$. Consider the following statements: P: $M^8 + M^{12}$ is diagonalizable. Q: $M^7 + M^9$ is diagonalizable. Which of the following statements is correct?
(A)	P is TRUE and Q is FALSE
(B)	P is FALSE and Q is TRUE
(C)	Both P and Q are FALSE
(D)	Both P and Q are TRUE

Q.20	Let $C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$. Consider the metric space $(C[0,1], d_\infty)$, where $$d_\infty(f,g) = \sup\{ \lvert f(x) - g(x)\rvert : x \in [0,1] \} \text{ for } f,g \in C[0,1].$$ Let $f_0(x) = 0$ for all $x \in [0,1]$ and $$X = \left\{ f \in (C[0,1], d_\infty) : d_\infty(f_0, f) \geq \frac{1}{2} \right\}.$$ Let $f_1, f_2 \in C[0,1]$ be defined by $f_1(x) = x$ and $f_2(x) = 1 - x$ for all $x \in [0,1]$. Consider the following statements: P: f_1 is in the interior of X. Q: f_2 is in the interior of X. Which of the following statements is correct?
(A)	P is TRUE and Q is FALSE
(B)	P is FALSE and Q is TRUE
(C)	Both P and Q are FALSE
(D)	Both P and Q are TRUE

Q.21	Consider the metrics ρ_1 and ρ_2 on $\mathbb{R}$, defined by $$\rho_1(x,y) = \lvert x-y\rvert \text{ and } \rho_2(x,y) = \begin{cases} 0, & if\ x = y \\ 1, & if\ x \neq y \end{cases}$$ Let $X = \{n \in \mathbb{N} : n \geq 3\}$ and $Y = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$. Define $f: X \cup Y \to \mathbb{R}$ by $f(x) = \begin{cases} 2, & if\ x \in X \\ 3, & if\ x \in Y \end{cases}$ Consider the following statements: P: The function $f: (X \cup Y, \rho_1) \to (\mathbb{R}, \rho_1)$ is uniformly continuous. Q: The function $f: (X \cup Y, \rho_2) \to (\mathbb{R}, \rho_1)$ is uniformly continuous. Then
(A)	P is TRUE and Q is FALSE
(B)	P is FALSE and Q is TRUE
(C)	both P and Q are FALSE
(D)	both P and Q are TRUE

Q.22	Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation and the null space of T be the subspace of $\mathbb{R}^4$ given by $\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 4x_1 + 3x_2 + 2x_3 + x_4 = 0\}$. If $Rank(T - 3I) = 3$, where I is the identity map on $\mathbb{R}^4$, then the minimal polynomial of T is
(A)	$x(x-3)$
(B)	$x(x-3)^3$
(C)	$x^3(x-3)$
(D)	$x^2(x-3)^2$

Q.23	Let $C[0,1]$ denote the set of all real valued continuous functions defined on $[0,1]$ and $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ for all $f \in C[0,1]$. Let $$X = \{ f \in C[0,1] : f(0) = f(1) = 0 \}.$$ Define $F : (C[0,1], \|\cdot\|_\infty) \to \mathbb{R}$ by $F(f) = \int_0^1 f(t)dt$ for all $f \in C[0,1]$. Denote $S_X = \{f \in X : \|f\|_\infty = 1\}$. Then the set $\{f \in X : F(f) = \|F\|\} \cap S_X$ has
(A)	NO element
(B)	exactly one element
(C)	exactly two elements
(D)	an infinite number of elements

Q.24	Let X and Y be two topological spaces. A continuous map $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is compact in X for every compact subset K of Y, where $f^{-1}(K)$ is defined by $f^{-1}(K) = \{x \in X : f(x) \in K\}$. Consider $\mathbb{R}$ with the usual topology. If $\mathbb{R} \setminus \{0\}$ has the subspace topology induced from $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ has the product topology, then which of the following maps is proper?
(A)	$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = x$
(B)	$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, y)$
(C)	$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x$
(D)	$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x^2 - y^2$

Q.25	Consider the following Linear Programming Problem **P**: $$\begin{aligned} &\text{Minimize } 3x_1 + 4x_2 \\ &\text{subject to } \quad x_1 - x_2 \le 1, \\ &\qquad x_1 + x_2 \ge 3, \\ &\qquad x_1 \ge 0, \ x_2 \ge 0. \end{aligned}$$ The optimal value of the problem **P** is ____________.

Q.26	Let $u(x,t)$ be the solution of $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,\ \ x \in (-\infty,\infty),\ \ t > 0,$$ $$u(x,0) = \sin x,\quad x \in (-\infty,\infty),$$ $$\frac{\partial u}{\partial t}(x,0) = \cos\ x,\quad x \in (-\infty,\infty),$$ for some positive real number c. Let the domain of dependence of the solution u at the point $P(3,2)$ be the line segment on the x-axis with end points Q and R. If the area of the triangle PQR is 8 square units, then the value of c^2 is ____________.

Q.27	Let $$\frac{z}{1-z-z^2} = \sum_{n=0}^{\infty} a_n z^n,\ \ a_n \in \mathbb{R}$$ for all z in some neighbourhood of 0 in $\mathbb{C}$. Then the value of $a_6 + a_5$ is equal to __________.

Q.28	Let $p(x) = x^3 - 2x + 2$. If $q(x)$ is the interpolating polynomial of degree less than or equal to 4 for the data

x	-2	-1	0	1	3
$q(x)$	$p(-2)$	$p(-1)$	2.5	$p(1)$	$p(3)$

,

then the value of $\frac{d^4q}{dx^4}$ at $x = 0$ is ______________.

Q.29	For a fixed $c \in \mathbb{R}$, let $\alpha = \int_0^2 (9x^2 - 5cx^4)dx$. If the value of $\int_0^2 (9x^2 - 5cx^4)dx$ obtained by using the Trapezoidal rule is equal to α, then the value of c is __________ *(rounded off to 2 decimal places)*.

Q.30	If for some $\alpha \in \mathbb{R}$, $$\int_1^4 \int_{-x}^{x} \frac{1}{x^2+y^2}\, dydx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sec\theta}^{\alpha\sec\theta} \frac{1}{r}\, drd\theta,$$ then the value of α equals ________.

Q.31	Let S be the portion of the plane $z = 2x + 2y - 100$ which lies inside the cylinder $x^2 + y^2 = 1$. If the surface area of S is $\alpha\pi$, then the value of α is equal to ____________.

Q.32	Let $L^2[-1,1] = \{f : [-1,1] \to \mathbb{R} : f$ is Lebesgue measurable and $\int_{-1}^{1} \lvert f(x)\rvert^2 dx < \infty\}$ and the norm $\lVert f\rVert_2 = \left(\int_{-1}^{1} \lvert f(x)\rvert^2\, dx\right)^{\frac{1}{2}}$ for $f \in L^2[-1,1]$. Let $F : (L^2[-1,1], \lVert\cdot\rVert_2) \to \mathbb{R}$ be defined by $$F(f) = \int_{-1}^{1} f(x)x^2 dx \text{ for all } f \in L^2[-1,1].$$ If $\lVert F\rVert$ denotes the norm of the linear functional F, then $5\lVert F\rVert^2$ is equal to ________.

Q.33	Let $y(t)$ be the solution of the initial value problem $$y'' + 4y = \begin{cases} t, & 0 \le t \le 2, \\ 2, & 2 < t < \infty, \end{cases} \text{ and } y(0) = y'(0) = 0.$$ If $\alpha = y\left(\frac{\pi}{2}\right)$, then the value of $\frac{4}{\pi}\alpha$ is ________ *(rounded off to 2 decimal places).*

Q.34	Consider $\mathbb{R}^4$ with the inner product $< x, y > = \sum_{i=1}^{4} x_i y_i$, for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. Let $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_3\}$ and $M^{\perp}$ denote the orthogonal complement of M. The dimension of $M^{\perp}$ is equal to ________.

Q.35	Let $M = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If $6M^{-1} = M^2 - 6M + \alpha I$ for some $\alpha \in \mathbb{R}$, then the value of α is equal to ___________.

Q.36 – Q.65 Carry TWO marks Each

Q.36	Let $GL_2(\mathbb{C})$ denote the group of 2×2 invertible complex matrices with usual matrix multiplication. For $S, T \in GL_2(\mathbb{C})$, $< S, T >$ denotes the subgroup generated by S and T. Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in GL_2(\mathbb{C})$ and G_1, G_2, G_3 be three subgroups of $GL_2(\mathbb{C})$ given by $G_1 = < S, T_1 >$, where $T_1 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $G_2 = < S, T_2 >$, where $T_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $G_3 = < S, T_3 >$, where $T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $Z(G_i)$ denote the center of G_i for $i = 1, 2, 3$. Which of the following statements is correct?
(A)	G_1 is isomorphic to G_3
(B)	$Z(G_1)$ is isomorphic to $Z(G_2)$
(C)	$Z(G_3) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
(D)	$Z(G_2)$ is isomorphic to $Z(G_3)$

Q.37	Let $\ell^2 = \{(x_1, x_2, x_3, \dots) : x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} x_n^2 < \infty\}$. For a sequence $(x_1, x_2, x_3, \dots) \in \ell^2$, define $\|(x_1, x_2, x_3, \dots)\|_2 = (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}}$. Let $S : (\ell^2, \|\cdot\|_2) \to (\ell^2, \|\cdot\|_2)$ and $T : (\ell^2, \|\cdot\|_2) \to (\ell^2, \|\cdot\|_2)$ be defined by $$S(x_1, x_2, x_3, \dots) = (y_1, y_2, y_3, \dots), \text{ where } y_n = \begin{cases} 0, & n = 1 \\ x_{n-1}, & n \geq 2 \end{cases}$$ $$T(x_1, x_2, x_3, \dots) = (y_1, y_2, y_3, \dots), \text{ where } y_n = \begin{cases} 0, & n \text{ is odd} \\ x_n, & n \text{ is even} \end{cases}$$ Then
(A)	S is a compact linear map and T is NOT a compact linear map
(B)	S is NOT a compact linear map and T is a compact linear map
(C)	both S and T are compact linear maps
(D)	NEITHER S NOR T is a compact linear map

Q.38	Let $c_{00} = \{(x_1, x_2, x_3, \dots) : x_i \in \mathbb{R}, i \in \mathbb{N}, x_i \neq 0$ only for finitely many indices $i\}$. For $(x_1, x_2, x_3, \dots) \in c_{00}$, let $\|(x_1, x_2, x_3, \dots)\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\}$. Define $F, G : (c_{00}, \|\cdot\|_\infty) \to (c_{00}, \|\cdot\|_\infty)$ by $F\big((x_1, x_2, \dots, x_n, \dots)\big) = \left((1+1)x_1, \left(2+\frac{1}{2}\right)x_2, \dots, \left(n+\frac{1}{n}\right)x_n, \dots\right)$, $G\big((x_1, x_2, \dots, x_n, \dots)\big) = \left(\frac{x_1}{1+1}, \frac{x_2}{2+\frac{1}{2}}, \dots, \frac{x_n}{n+\frac{1}{n}}, \dots\right)$, for all $(x_1, x_2, \dots, x_n, \dots) \in c_{00}$. Then
(A)	F is continuous but G is NOT continuous
(B)	F is NOT continuous but G is continuous
(C)	both F and G are continuous
(D)	NEITHER F NOR G is continuous

Q.39	Consider the Cauchy problem $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u;$$ $u=f(t)$ on the initial curve $\Gamma=(t,\ t);\ t>0$. Consider the following statements: P: If $f(t)=\ 2t\ +1$, then there exists a unique solution to the Cauchy problem in a neighbourhood of Γ. Q: If $f(t)=\ 2t-1$, then there exist infinitely many solutions to the Cauchy problem in a neighbourhood of Γ. Then
(A)	both P and Q are TRUE
(B)	P is FALSE and Q is TRUE
(C)	P is TRUE and Q is FALSE
(D)	both P and Q are FALSE

Q.40	Consider the linear system $Mx = b$, where $M = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Suppose $M = LU$, where L and U are lower triangular and upper triangular square matrices, respectively. Consider the following statements: P: If each element of the main diagonal of L is 1, then $trace(U) = 3$. Q: For any choice of the initial vector $x^{(0)}$, the Jacobi iterates $x^{(k)}, k = 1,2,3 \ldots$ converge to the unique solution of the linear system $Mx = b$. Then
(A)	both P and Q are TRUE
(B)	P is FALSE and Q is TRUE
(C)	P is TRUE and Q is FALSE
(D)	both P and Q are FALSE

Q.41	Let ϕ and ψ be two linearly independent solutions of the ordinary differential equation $$y'' + (2 - \cos x)\, y = 0, \qquad x \in \mathbb{R}.$$ Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, $\phi(\alpha) = \phi(\beta) = 0$ and $\phi(x) \neq 0$ for all $x \in (\alpha, \beta)$. Consider the following statements: P: $\phi'(\alpha)\phi'(\beta) > 0$. Q: $\phi(x)\psi(x) \neq 0$ for all $x \in (\alpha, \beta)$. Then
(A)	P is TRUE and Q is FALSE
(B)	P is FALSE and Q is TRUE
(C)	both P and Q are FALSE
(D)	both P and Q are TRUE

Q.42	Let $\mathbb{D} = \{ z \in \mathbb{C} : \ \|z\| < 1\}$ and $f : \mathbb{D} \to \mathbb{C}$ be an analytic function given by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_0 = a_1 = 1$ and $a_n = \frac{1}{2^{2n}}$ for $n \geq 2$. Consider the following statements: P: If $z_0 \in \mathbb{D}$, then f is one-one in some neighbourhood of z_0. Q: If $E = \left\{ z \in \mathbb{C} : \|z\| \leq \frac{1}{2} \right\}$, then $f(E)$ is a closed subset of $\mathbb{C}$. Which of the following statements is/are correct?
(A)	P is TRUE
(B)	Q is TRUE
(C)	Q is FALSE
(D)	P is FALSE

Q.43	Let Ω be an open connected subset of $\mathbb{C}$ containing $U = \left\{ z \in \mathbb{C} : \lvert z \rvert \le \frac{1}{2} \right\}$. Let $\mathfrak{I} = \{ f : \Omega \to \mathbb{C} : f \text{ is analytic and } \sup\limits_{z,w \in U} \lvert f(z) - f(w) \rvert = 1 \}$. Consider the following statements: P: There exists $f \in \mathfrak{I}$ such that $\lvert f'(0) \rvert \ge 2$. Q: $\left\lvert f^{(3)}(0) \right\rvert \le 48$ for all $f \in \mathfrak{I}$, where $f^{(3)}$ denotes the third derivative of f. Then
(A)	P is TRUE
(B)	Q is FALSE
(C)	P is FALSE
(D)	Q is TRUE

Q.44	Let $(\mathbb{R}, \tau)$ be a topological space, where the topology τ is defined as $$\tau = \{U \subseteq \mathbb{R} : U = \emptyset \ or \ 1 \in U\}.$$ Which of the following statements is/are correct?
(A)	$(\mathbb{R}, \tau)$ is first countable
(B)	$(\mathbb{R}, \tau)$ is Hausdorff
(C)	$(\mathbb{R}, \tau)$ is separable
(D)	The closure of $(1,5)$ is $[1,5]$

Q.45	Let $\mathcal{R} = \{p(x) \in \mathbb{Q}[x] : \; p(0) \in \mathbb{Z}\}$, where $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{Z}$ denotes the set of integers. For $a \in \mathcal{R}$, let $\langle a \rangle$ denote the ideal generated by a in $\mathcal{R}$. Which of the following statements is/are correct?
(A)	If $p(x)$ is an irreducible element in $\mathcal{R}$, then $\langle p(x) \rangle$ is a prime ideal in $\mathcal{R}$
(B)	$\mathcal{R}$ is a unique factorization domain
(C)	$\langle x \rangle$ is a prime ideal in $\mathcal{R}$
(D)	$\mathcal{R}$ is NOT a principal ideal domain

Q.46	Consider the rings $$\mathcal{S}_1 = \mathbb{Z}[x]/\langle 2, x^3\rangle \quad \text{and} \quad \mathcal{S}_2 = \mathbb{Z}_2[x]/\langle x^2\rangle$$ where $\langle 2, x^3\rangle$ denotes the ideal generated by $\{2, x^3\}$ in $\mathbb{Z}[x]$ and $\langle x^2\rangle$ denotes the ideal generated by x^2 in $\mathbb{Z}_2[x]$. Which of the following statements is/are correct?
(A)	Every prime ideal of $\mathcal{S}_1$ is a maximal ideal
(B)	$\mathcal{S}_2$ has exactly one maximal ideal
(C)	Every element of $\mathcal{S}_1$ is either nilpotent or a unit
(D)	There exists an element in $\mathcal{S}_2$ which is NEITHER nilpotent NOR a unit

Q.47	Consider the sequence of Lebesgue measurable functions $f_n: \mathbb{R} \to \mathbb{R}$ given by $$f_n(x) = \begin{cases} n^2(x-n), & if\ x \in \left[n, n+\frac{1}{n^2}\right] \\ 0, & otherwise \end{cases}$$ For a measurable subset E of $\mathbb{R}$, denote $m(E)$ to be the Lebesgue measure of E. Which of the following statements is/are correct?
(A)	$\sup_{x \in \mathbb{R}} \lvert f_n(x) \rvert \to 0\ as\ n \to \infty$
(B)	$\int_{\mathbb{R}} \lvert f_n(x) \rvert dx \to 0\ as\ n \to \infty$
(C)	$m\left(\left\{x \in \mathbb{R} : \lvert f_n(x) \rvert > \frac{1}{2}\right\}\right) \to 0\ as\ n \to \infty$
(D)	$m(\{x \in \mathbb{R} : \lvert f_n(x) \rvert > 0\}) \to 0\ as\ n \to \infty$

Q.48	Define the characteristic function χ_E of a subset E in $\mathbb{R}$ by $$\chi_E(x) = \begin{cases} 1, & if\ x \in E \\ 0, & if\ x \notin E \end{cases}$$ For $1 \leq p < 2$, let $L^p[0,1] = \{f:[0,1] \to \mathbb{R} : f$ is Lebesgue measurable and $\int_0^1 \lvert f(x)\rvert^p dx < \infty\}$. Let $f:[0,1] \to \mathbb{R}$ be defined by $$f(x) = \sum_{n=1}^{\infty} \frac{2^n}{n^3} \chi_{\left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]}(x).$$ Consider the following two statements: P: $f \in L^p[0,1]$ for every $p \in (1,2)$. Q: $f \in L^1[0,1]$. Then
(A)	P is TRUE
(B)	Q is TRUE
(C)	Q is FALSE
(D)	P is FALSE

Q.49	Let $x(t),\ y(t),\ t \in \mathbb{R}$, be two functions satisfying the following system of differential equations: $$x'(t) = y(t),$$ $$y'(t) = x(t),$$ and $x(0) = \alpha,\ y(0) = \beta$, where α, β are real numbers. Which of the following statements is/are correct?
(A)	If $\alpha = 1,\ \beta = -1,$ then $\lvert x(t)\rvert + \lvert y(t)\rvert \to 0\ as\ t \to \infty$
(B)	If $\alpha = 1,\ \beta = 1,$ then $\lvert x(t)\rvert + \lvert y(t)\rvert \to 0\ as\ t \to \infty$
(C)	If $\alpha = 1.01,\ \beta = -1,$ then $\lvert x(t)\rvert + \lvert y(t)\rvert \to 0\ as\ t \to \infty$
(D)	If $\alpha = 1,\ \beta = 1.01,$ then $\lvert x(t)\rvert + \lvert y(t)\rvert \to 0\ as\ t \to \infty$

Q.50	For $h > 0$, and $\alpha, \beta, \gamma \in \mathbb{R}$, let $$D_h f(a) = \frac{\alpha f(a-h) + \beta f(a) + \gamma f(a+2h)}{6h}$$ be a three-point formula to approximate $f'(a)$ for any differentiable function $f: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$. If $D_h f(a) = f'(a)$ for every polynomial f of degree less than or equal to 2 and for all $a \in \mathbb{R}$, then
(A)	$\alpha + 2\gamma = -2$
(B)	$\alpha + 2\beta - 2\gamma = 0$
(C)	$\alpha + 2\gamma = 2$
(D)	$\alpha + 2\beta - 2\gamma = 1$

Q.51	Let f be a twice continuously differentiable function on $[a, b]$ such that $f'(x) < 0$ and $f''(x) < 0$ for all $x \in (a, b)$. Let $f(\zeta) = 0$ for some $\zeta \in (a, b)$. The Newton-Raphson method to compute ζ is given by $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots$$ for an initial guess x_0. If $x_k \in (\zeta, b)$ for some $k \geq 0$, then which of the following statements is/are correct?
(A)	$x_{k+1} > \zeta$
(B)	$x_{k+1} < \zeta$
(C)	$x_{k+1} < x_k$
(D)	For every $\eta \in (\zeta, x_k)$, $\frac{f'(\eta)}{f'(x_k)} > 1$

Q.52	Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$ Then
(A)	the directional derivative of f at (0,0) in the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $\frac{1}{\sqrt{2}}$
(B)	the directional derivative of f at (0,0) in the direction of $(0, 1)$ is 1
(C)	the directional derivative of f at (0,0) in the direction of $(1, 0)$ is 0
(D)	f is NOT differentiable at $(0,0)$

Q.53	Let $C[0,1] = \{ f: [0,1] \to \mathbb{R} \ : \ f \text{ is continuous}\}$ and $d_\infty(f,g) = \sup\{ \lvert f(x) - g(x) \rvert : x \in [0,\ 1]\}$ for $f, g \in C[0,1]$. For each $n \in \mathbb{N}$, define $f_n: [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$ for all $x \in [0,1]$. Let $P = \{f_n:\ n \in \mathbb{N}\}$. Which of the following statements is/are correct?
(A)	P is totally bounded in $(C[0,1],\ d_\infty)$
(B)	P is bounded in $(C[0,1],\ d_\infty)$
(C)	P is closed in $(C[0,1],\ d_\infty)$
(D)	P is open in $(C[0,1],\ d_\infty)$

Q.54	Let G be an abelian group and $\Phi : G \to (\mathbb{Z}, +)$ be a surjective group homomorphism. Let $1 = \Phi(a)$ for some $a \in G$. Consider the following statements: P: For every $g \in G$, there exists an $n \in \mathbb{Z}$ such that $ga^n \in \ker(\Phi)$. Q: Let e be the identity of G and $< a >$ be the subgroup generated by a. Then $G = \ker(\Phi) < a >$ and $\ker(\Phi) \cap < a > = \{e\}$. Which of the following statements is/are correct?
(A)	P is TRUE
(B)	P is FALSE
(C)	Q is TRUE
(D)	Q is FALSE

Q.55	Let C be the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z - 2 = 0$. Suppose C is oriented in the counterclockwise direction around the z-axis, when viewed from above. If $$\left\| \int_C (\sin x + e^x)\, dx + 4x\, dy + e^z \cos^2 z\, dz \right\| = \alpha\pi,$$ then the value of α equals ____________.

Q.56	Let $\ell^2 = \{(x_1, x_2, x_3, \ldots) : x_n \in \mathbb{R}\ for\ all\ n \in \mathbb{N}\ and\ \sum_{n=1}^{\infty} x_n^2 < \infty\}$. For a sequence $(x_1, x_2, x_3, \ldots) \in \ell^2$, define $$\|(x_1, x_2, x_3, \ldots)\|_2 = \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$$ Consider the subspace $M = \left\{(x_1, x_2, x_3, \ldots) \in \ell^2 : \sum_{n=1}^{\infty} \frac{x_n}{4^n} = 0\right\}$. Let $M^{\perp}$ denote the orthogonal complement of M in the Hilbert space $(\ell^2, \|\cdot\|_2)$. Consider $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) \in \ell^2$. If the orthogonal projection of $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ onto $M^{\perp}$ is given by $\alpha\left(\sum_{n=1}^{\infty} \frac{1}{n4^n}\right)\left(\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots\right)$ for some $\alpha \in \mathbb{R}$, then α equals __________.

Q.57	Consider the transportation problem between five sources and four destinations as given in the cost table below. The supply and demand at each of the source and destination are also provided:

		DESTINATIONS				**Supply**
		P	**Q**	**R**	**S**	
SOURCES	**1**	13	8	12	9	20
	2	10	7	5	20	10
	3	3	19	5	12	50
	4	4	9	7	15	30
	5	14	0	1	7	40
Demand		60	10	20	60	

Let C_N and C_L be the total cost of the initial basic feasible solution obtained from the North-West corner method and the Least-Cost method, respectively. Then $C_N - C_L$ equals __________.

Q.58	Let $\sigma \in S_8$, where S_8 is the permutation group on 8 elements. Suppose σ is the product of σ_1 and σ_2, where σ_1 is a 4-cycle and σ_2 is a 3-cycle in S_8. If σ_1 and σ_2 are disjoint cycles, then the number of elements in S_8 which are conjugate to σ is __________.

Q.59	Let A be a 3×3 real matrix with $\det(A + i\,I) = 0$, where $i = \sqrt{-1}$ and I is the 3×3 identity matrix. If $\det(A) = 3$, then the trace of A^2 is ___________.

Q.60	Let $A = [a_{ij}]$ be a 3×3 real matrix such that $$A\begin{bmatrix}1\\2\\1\end{bmatrix} = 2\begin{bmatrix}1\\2\\1\end{bmatrix},\ A\begin{bmatrix}0\\1\\1\end{bmatrix} = 2\begin{bmatrix}0\\1\\1\end{bmatrix} \text{ and } A\begin{bmatrix}-1\\1\\0\end{bmatrix} = 4\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$ If m is the degree of the minimal polynomial of A, then $a_{11} + a_{21} + a_{31} + m$ equals ___________.

Q.61	Let Ω be the disk $x^2 + y^2 < 4$ in $\mathbb{R}^2$ with boundary $\partial\Omega$. If $u(x, y)$ is the solution of the Dirichlet problem $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega,$$ $$u(x, y) = 1 + 2\,x^2, \ \ (x, y) \in \partial\Omega,$$ then the value of $u(0,1)$ is ____________.

Q.62	For every $k \in \mathbb{N} \cup \{0\}$, let $y_k(x)$ be a polynomial of degree k with $y_k(1) = 5$. Further, let $y_k(x)$ satisfy the Legendre equation $$(1 - x^2)y'' - 2xy' + k(k+1)y = 0.$$ If $$\frac{1}{2}\int_{-1}^{1}\sum_{k=1}^{n}\big(y_k(x) - y_{k-1}(x)\big)^2\,dx - \int_{-1}^{1}\sum_{k=1}^{n}\big(y_k(x)\big)^2\,dx = 24,$$ for some positive integer n, then the value of n is ____________.

Q.63	Consider the ordinary differential equation (ODE) $$4\,(\ln x)\,y'' + 3\,y' + y = 0, \qquad x > 1.$$ If r_1 and r_2 are the roots of the indicial equation of the above ODE at the regular singular point $x = 1$, then $\lvert r_1 - r_2 \rvert$ is equal to ________ *(rounded off to 2 decimal places).*

Q.64	Let $u(x,t)$ be the solution of the non-homogeneous wave equation $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin x \sin(2t), \qquad 0 < x < \pi,\ \ t > 0$$ $$u(x,0) = 0,\ \ and\ \frac{\partial u}{\partial t}(x,0) = 0, \qquad for\ 0 \le x \le \pi,$$ $$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad for\ t \ge 0.$$ Then the value of $u\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ is ___________ *(rounded off to 2 decimal places).*

Q.65	Consider the Linear Programming Problem **P**: $$\text{Maximize } 3x_1 + 2x_2 + 5x_3$$ subject to $$x_1 + 2x_2 + x_3 \leq 44,$$ $$x_1 + 2x_3 \leq 48,$$ $$x_1 + 4x_2 \leq 52,$$ $$x_1 \geq 0,\ \ x_2 \geq 0,\ \ x_3 \geq 0.$$ The optimal value of the problem **P** is equal to ____________.

END OF QUESTION PAPER