Q.1 Suppose \(a, b\) denote the distinct real roots of the quadratic polynomial \(x^2 + 20x - 2020\) and suppose \(c, d\) denote the distinct complex roots of the quadratic polynomial \(x^2 - 20x + 2020\). Then the value of

\[ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)\]

is

(A) 0  
(B) 8000  
(C) 8080  
(D) 16000

Q.2 If the function \(f: \mathbb{R} \rightarrow \mathbb{R}\) is defined by \(f(x) = |x|(x - \sin x)\), then which of the following statements is TRUE?

(A) \(f\) is one-one, but NOT onto  
(B) \(f\) is onto, but NOT one-one  
(C) \(f\) is BOTH one-one and onto  
(D) \(f\) is NEITHER one-one NOR onto

Q.3 Let the functions \(f: \mathbb{R} \rightarrow \mathbb{R}\) and \(g: \mathbb{R} \rightarrow \mathbb{R}\) be defined by

\[f(x) = e^{x-1} - e^{-|x-1|}\] \[g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).\]

Then the area of the region in the first quadrant bounded by the curves \(y = f(x), y = g(x)\) and \(x = 0\) is

(A) \((2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})\)  
(B) \((2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})\)  
(C) \((2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})\)  
(D) \((2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})\)
Q.4 Let $a, b$ and $\lambda$ be positive real numbers. Suppose $P$ is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point $P$. If the tangents to the parabola and the ellipse at the point $P$ are perpendicular to each other, then the eccentricity of the ellipse is

\[
\begin{align*}
(A) & \quad \frac{1}{\sqrt{2}} \\
(B) & \quad \frac{1}{2} \\
(C) & \quad \frac{1}{3} \\
(D) & \quad \frac{2}{5}
\end{align*}
\]

Q.5 Let $C_1$ and $C_2$ be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose $\alpha$ is the number of heads that appear when $C_1$ is tossed twice, independently, and suppose $\beta$ is the number of heads that appear when $C_2$ is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

\[
\begin{align*}
(A) & \quad \frac{40}{81} \\
(B) & \quad \frac{20}{81} \\
(C) & \quad \frac{1}{2} \\
(D) & \quad \frac{1}{4}
\end{align*}
\]

Q.6 Consider all rectangles lying in the region

\[
\{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2 \sin(2x)\}
\]

and having one side on the $x$-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

\[
\begin{align*}
(A) & \quad \frac{3\pi}{2} \\
(B) & \quad \pi \\
(C) & \quad \frac{\pi}{2\sqrt{3}} \\
(D) & \quad \frac{\pi\sqrt{3}}{2}
\end{align*}
\]
Q.7 Let the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = x^3 - x^2 + (x - 1) \sin x \) and let \( g: \mathbb{R} \rightarrow \mathbb{R} \) be an arbitrary function. Let \( fg: \mathbb{R} \rightarrow \mathbb{R} \) be the product function defined by \((fg)(x) = f(x)g(x)\). Then which of the following statements is/are TRUE?

(A) If \( g \) is continuous at \( x = 1 \), then \( fg \) is differentiable at \( x = 1 \)

(B) If \( fg \) is differentiable at \( x = 1 \), then \( g \) is continuous at \( x = 1 \)

(C) If \( g \) is differentiable at \( x = 1 \), then \( fg \) is differentiable at \( x = 1 \)

(D) If \( fg \) is differentiable at \( x = 1 \), then \( g \) is differentiable at \( x = 1 \)

Q.8 Let \( M \) be a \( 3 \times 3 \) invertible matrix with real entries and let \( I \) denote the \( 3 \times 3 \) identity matrix. If \( M^{-1} = \text{adj} (\text{adj} M) \), then which of the following statements is/are ALWAYS TRUE?

(A) \( M = I \)

(B) \( \det M = 1 \)

(C) \( M^2 = I \)

(D) \( (\text{adj} M)^2 = I \)

Q.9 Let \( S \) be the set of all complex numbers \( z \) satisfying \( |z^2 + z + 1| = 1 \). Then which of the following statements is/are TRUE?

(A) \( |z + \frac{1}{2}| \leq \frac{1}{2} \) for all \( z \in S \)

(B) \( |z| \leq 2 \) for all \( z \in S \)

(C) \( |z + \frac{1}{2}| \geq \frac{1}{2} \) for all \( z \in S \)

(D) The set \( S \) has exactly four elements

Q.10 Let \( x, y \) and \( z \) be positive real numbers. Suppose \( x, y \) and \( z \) are the lengths of the sides of a triangle opposite to its angles \( X, Y \) and \( Z \), respectively. If

\[
\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z},
\]

then which of the following statements is/are TRUE?

(A) \( 2Y = X + Z \)

(B) \( Y = X + Z \)

(C) \( \tan \frac{x}{2} = \frac{x}{y+z} \)

(D) \( x^2 + z^2 - y^2 = xz \)
Q.11 Let $L_1$ and $L_2$ be the following straight lines.

$$L_1: \frac{x - 1}{1} = \frac{y}{-1} = \frac{z - 1}{3} \quad \text{and} \quad L_2: \frac{x - 1}{-3} = \frac{y}{-1} = \frac{z - 1}{1}.$$ 

Suppose the straight line

$$L: \frac{x - \alpha}{l} = \frac{y - 1}{m} = \frac{z - \gamma}{-2}$$

lies in the plane containing $L_1$ and $L_2$, and passes through the point of intersection of $L_1$ and $L_2$. If the line $L$ bisects the acute angle between the lines $L_1$ and $L_2$, then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$  
(B) $l + m = 2$  
(C) $\alpha - \gamma = 1$  
(D) $l + m = 0$

Q.12 Which of the following inequalities is/are TRUE?

(A) $\int_0^1 x \cos x \, dx \geq \frac{3}{8}$  
(B) $\int_0^1 x \sin x \, dx \geq \frac{3}{10}$  
(C) $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$  
(D) $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{5}$

Q.13 Let $m$ be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where $y_1$, $y_2$, $y_3$ are real numbers for which $y_1 + y_2 + y_3 = 9$. Let $M$ be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where $x_1$, $x_2$, $x_3$ are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____.
Q.14 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of $c$, for which the equality
\[2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n\]
holds for some positive integer $n$, is _____

Q.15 Let $f : [0, 2] \rightarrow \mathbb{R}$ be the function defined by
\[f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin(3\pi x + \frac{\pi}{4}).\]
If $\alpha, \beta \in [0, 2]$ are such that \(\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]\), then the value of $\beta - \alpha$ is _____

Q.16 In a triangle $PQR$, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If
\[|\vec{a}| = 3, \quad |\vec{b}| = 4 \quad \text{and} \quad \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|},\]
then the value of $|\vec{a} \times \vec{b}|^2$ is _____

Q.17 For a polynomial $g(x)$ with real coefficients, let $m_g$ denote the number of distinct real roots of $g(x)$. Suppose $S$ is the set of polynomials with real coefficients defined by
\[S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.
For a polynomial $f$, let $f'$ and $f''$ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Q.18 Let $e$ denote the base of the natural logarithm. The value of the real number $a$ for which the right hand limit
\[\lim_{x \to 0^+} \frac{(1 - x)^{\frac{1}{x}} - e^{-1}}{x^a}\]
is equal to a nonzero real number, is _____

END OF THE QUESTION PAPER